

Managing Appointment Scheduling under Patient Choices

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Abstract

Motivated by the use of appointment templates in healthcare scheduling practice, we study how to offer appointment slots to patients in order to maximize the utilization of provider time. We develop two models, *non-sequential* scheduling and *sequential* scheduling, to capture different types of interactions between patients and the scheduling system. The scheduler offers either a single set of appointment slots for the arriving patient to choose from, or multiple sets in sequence, respectively. This is done without knowledge of patient preference information. For the non-sequential scheduling model, we identify certain problem instances where the greedy policy (i.e., offering all available slots throughout) is suboptimal, but show through analytical and numerical results that for most moderate and large instances greedy performs remarkably well. For the sequential model we derive the optimal offering policy for a large class of instances, and develop an effective and simple-to-use heuristic inspired by fluid models. We present a case study based on real patient preference data, and demonstrate a potential improvement of up to 17% in provider capacity utilization by adopting our proposed scheduling policies. This improvement may translate into \$45k-\$120k increase in annual revenues for a single primary care provider.

keywords service operations; healthcare operations management; patient choices; appointment scheduling; Markov decision processes

1 Introduction

Outpatient care providers often manage their services by scheduling appointments ahead of time. They first predetermine for each day an appointment template (also called master schedule), which specifies the total number of slots, the length of each slot, and in some cases the number¹ and characteristics of patients to be scheduled for each slot. With an appointment template in place, these providers then decide how to assign incoming patient requests to the available slots. The relevant performance metric for this process is the *fill rate*, i.e., the fraction of slots in a template booked before the scheduling process closes.

The importance of ensuring a high fill rate cannot be overstated in current healthcare practice. With the world-wide population aging, the imbalance between rising patient demand and limited health service capacity is a global problem (Dall et al. 2013). In the United States, the Patient Protection and Affordable Care Act expands health insurance coverage, and increases patient volumes and visits significantly; surging patient demand is likely to exacerbate the existing access problem faced by many patients due to the long-standing primary care workforce shortage (Bodenheimer and Pham 2010). While the fill rate is not equivalent to the eventual capacity utilization due to various post-scheduling factors (e.g., late cancellations, patient no-shows and walk-ins), it is the first, and in many cases, the most important step to achieving a high utilization, and it is the objective of the research presented in this paper.

Our focus is on modeling the scheduling process, and developing stochastic dynamic optimization models for scheduling appointment requests in the presence of patient choice behavior. Notwithstanding the surge

¹For instance, the classic Bailey-Welch rule indicates that the first appointment slot in a clinic session should be overbooked with two patients (Welch and Bailey 1952) to prevent doctor idling.

of interest in healthcare operations research in the past decade, basic single-day, choice-based dynamic decision models are absent for a broad class of real-world scheduling systems. To our knowledge, the existing operations research literature on this type of dynamic appointment scheduling consists of only three papers: Gupta and Wang (2008), Wang and Gupta (2011), and Wang and Fung (2015). They all assume that patients reveal their preference first and the scheduler decides to accept or reject. This may not be representative for all patient scheduling systems in practice, however.

Instead, for a broad class of real-world scheduling systems, the scheduler offers its availability to the patient to choose from. This includes increasingly popular web and mobile-based online appointment booking systems such as www.zocdoc.com and www.questdiagnostics.com, as well as the conventional telephone-based scheduling process, conducted through conversations between an agent and the patient. This paper fills a gap in the appointment scheduling literature by proposing the first choice-based dynamic optimization models for scheduling decisions in systems where patients are allowed to choose among offered appointment slots from an established appointment template. We demonstrate how the current practice can be improved by developing optimality results and heuristics in the context of the proposed models.

We propose and study two models for the interaction between patients and the provider. The first one is referred to as the *non-sequential offering* model. In this model, the scheduler offers a single set of appointment slots to each patient. If some of these offered slots are acceptable to the patient, she chooses one from them; otherwise, the patient does not book an appointment. This simple, one-time interaction resembles the mechanism of many web-based online appointment systems and our analytical results on this first model have direct implications on how to manage these systems. Our second model is a *sequential offering* model, in which the scheduler may offer several sets of appointment choices in a sequential manner. This is motivated by 1) mobile-based appointment applications, in which the patient usually cannot view all available slots in one screen display due to the small size of mobile devices; and 2) the telephone-based scheduling process, in which the scheduling agent may reveal availability of appointment slots in a sequential manner. This second model is stylized, as it does not incorporate patient recall behavior (i.e., a patient choosing a previously offered slot after viewing more offers), which is allowed in both mobile- and phone-based applications. Our goal here is to glean insights on how the fill rate can be improved by “smarter” sequencing when sequential offering is part of the scheduling process.

For both cases we are interested in the question which slots to offer in order to improve the fill rate. We answer this question by investigating the optimal offering policy using Markov decision processes, as well as by discussing heuristics. In the non-sequential setting we demonstrate that in many cases the greedy policy, which offers all available slots, is in fact asymptotically optimal as the numbers of slot types and patient types grow large, and does very well in the vast majority of problem instances. We identify a small set of instances where greedy can be improved upon, and derive optimal offering policies that lead to a sizable 3-6% improvement in the fill-rate over the greedy policy. In the sequential case the greedy policy does not do well, but we fully characterize the optimal offering policy for a large class of problem instances. In some cases, the optimal sequential offering policy can result in up to 17% improvement compared to the greedy policy. In addition, we develop an easy-to-implement heuristic, which is numerically shown to perform remarkably well. Our models and heuristics are tested in a case study based on realistic patient choice parameters estimated from an online survey study; this case study reveals significant efficiency improvement over current practice.

In summary, we make the following contributions to the literature:

- To the best of our knowledge, our paper is the first to consider dynamic offering of appointment slots while explicitly taking into account heterogeneous patient preferences. Our models and results are supported by experiments based on data collected via patient preference surveys and are relevant to the two main scheduling paradigms, non-sequential (online) and sequential (telephone-based), used in the healthcare market.
- We present operational insights and heuristics for patient scheduling practice, and show an improvement of up to 17% compared to current practice. Given the small margins in healthcare industries (Langabeer 2007), this level of improvement can significantly increase the performance and financial sustainability of an organization.
- This paper considers a natural yet novel choice model for the application of healthcare appointment scheduling. The parsimonious choice model is based on the notion of patient and slot types, and as

shown in Section 5.1, the model may be easily estimated by straightforward surveys. Given the limited resource and transaction data in healthcare (Viney et al. 2002), such simple estimation is essential.

It is worth pointing out that our work is complementary to the prior literature on the design of appointment templates (see a brief review below), in that we start from an established template, and then study how to manage the interaction between the patients and the scheduler in order to best allocate patients to various slots. This “allocation” problem we consider is essential and non-trivial, as different patients may have different slot preferences. The more popular slots provide the scheduler with more flexibility, but they may be filled too early in the booking process and thus it requires careful “rationing.”

The remainder of the paper is organized as follows. Section 1.1 reviews the relevant literature. Section 2 introduces the common capacity and demand model that will be used in both the non-sequential and sequential models. Sections 3 and 4 discuss the non-sequential offering case and the sequential offering case, respectively. We present some evidence on patient choice behavior via our survey study in Section 5.1, and use the results of this study to test our scheduling policies in Section 5.2. In Section 6, we discuss managerial insights from our analytic work and make concluding remarks. All proofs of our technical results can be found in the online appendix.

1.1 Literature Review

From an application perspective, our work is related and complementary to the literature on appointment template design, a topic that has been studied extensively. See, for instance, Cayirli and Veral (2003), Gupta and Denton (2008) and Ahmadi-Javid et al. (2016) for in-depth reviews of this literature. As for the existing work on dynamic appointment scheduling, Feldman et al. (2014) is the only study, other than the three papers mentioned in the previous section, that explicitly models patient choice behavior. However, Feldman et al. (2014) focus on patient choices across different days and use a newsvendor-like model to capture the use of daily capacity; this aggregate daily capacity model does not allow them to consider (allocating patients into) detailed appointment time slots within a daily template, as is common in practice.

From a modeling perspective, Zhang and Cooper (2005) looks at a similar choice model to ours, in the context of revenue management for parallel flights. In contrast to the present paper, their approach focuses on deriving bounds on the value function of the underlying MDP, and using them to construct heuristics. Three recent studies on assortment optimization are particularly relevant to our paper: Bernstein et al. (2015), Golrezaei et al. (2014) and Gallego et al. (2016). Bernstein et al. (2015) study a dynamic assortment customization problem, mathematically similar to our non-sequential appointment offering problem, assuming multiple types of customers, each of which has a multinomial logit choice behavior over *all* product types. In particular, although a customer type may choose a product type with an arbitrarily small probability in their model, that probability is always positive, whereas we assume a binary choice model in which a patient either accepts or does not accept a slot at all. They assume that the customer type is observable to the seller (corresponding to our scheduler), which differs from our setting. Golrezaei et al. (2014) adopt a general choice model that covers both our binary form choice model and the multinomial logit model used by Bernstein et al. (2015). They also allow an arbitrary customer arrival process, which covers the stochastic independent and identically distributed arrival process assumed in both our work and Bernstein et al. (2015). Gallego et al. (2016) extend the work by Golrezaei et al. (2014) to allow rewards that depend on both the customer type and product type. Yet, the last two studies assume that the customer type is known to the seller, and their focus is on developing control policies competitive with respect to an offline optimum, a different type of research question from ours. The other distinguishing feature of our research from all previous work is that we consider sequential offering, an offering paradigm which has not been studied before.

Another relevant work to ours is Akçay et al. (2010), who study dynamic assignment of flexible service resources. Similar to our choice model (discussed below), they use a bipartite graph to model the flexibility of various resources. In their setting, the manager needs to decide whether to accept incoming jobs and what resource to assign to each accepted job. Their model is quite general in terms of the reward structure. However, they do not consider customer choice, and assume that the manager has full knowledge of the customer “type.”

Finally, our work is related to two other branches of literature. The first on online bipartite matching (see Mehta 2013, Mehta and Panigrahi 2012, Mehta et al. 2015), and the second on general stochastic dynamic optimization, in particular stochastic depletion problems (see Chan and Farias 2009 and the references

therein) and submodular optimization (see Golovin and Krause 2011 and the references therein). These two lines of research mainly aim to obtain performance guarantee results with respect to offline optimums, which is not our research goal.

2 Capacity and Demand Model

We consider a single day in the future that has just opened for appointment booking. The day has an established appointment template, but none of the slots are filled yet. We divide the appointment scheduling window, i.e., the time between when the day is first opened for booking and the end time of this booking process, into N small periods. Specifically, we consider a discrete-time N -period dynamic optimization model with I patient types (that may come) and J slot types (in the template), where patient types are characterized by their set of *acceptable* appointment slot types. Denote by Ω_{ij} the 0-1 indicator of whether slot type j is acceptable by patient type i , so the $I \times J$ *choice matrix* $\Omega := [\Omega_{ij}]$ consists of distinct row vectors, each representing a unique patient type.

We now present the details of our patient arrival and choice model. The choice model is corroborated by a survey study presented in Section 5.1. In each period at most one patient arrives: The patient is type i with probability $\lambda_i > 0$, and with probability $\lambda_0 := 1 - \sum_{i=1}^I \lambda_i$ no patient arrives. Upon patient arrival, the scheduler offers her a set $S \subseteq \{1, \dots, J\}$ of slot types, *without* knowledge of the patient type. When *offer set* S contains one or more acceptable slot types, the patient chooses one uniformly at random. If no type in S is acceptable to this patient, we distinguish two possibilities. Either we use a *non-sequential* model where the scheduler can only offer a single set, and the patient immediately leaves without selecting a slot (see Section 3), or we use a *sequential* model where the scheduler may offer any number of sets sequentially, until the patient either encounters an acceptable slot, or the patient finds no acceptable slots in any offer set and leaves without selecting a slot (see Section 4). We start from an initial capacity of b_j slots of type j at the beginning of the reservation process, and denote $\mathbf{b} := (b_1, \dots, b_J)$. Every time a patient selects a slot, the remaining number of slots of this type is reduced by 1. The scheduler aims to maximize the fill rate at the end of the reservation process by deciding on the offer set(s) in each period. This is also equivalent to maximizing the *fill count*, i.e., the total number of slots reserved at the end of the booking process, because the initial capacity \mathbf{b} is fixed.

Our capacity and demand model generalizes that of Wang and Gupta (2011) in the following sense. Our notion of ‘slot type’ can be viewed as an abstraction of the (physician, time block) combination in their model, and thus we allow a generalization of using other attributes of a slot that may affect its acceptability to patients, such as duration. Wang and Gupta (2011) consider distinct patient panels, each characterized by a possibly different acceptance probability distribution over all possible combinations of physicians and time blocks and a set of revenue parameters. In contrast, we define the notion of a patient type and identify it with a unique set of acceptable slot types. Their arrival rate (probability) parameters are associated with each patient panel, while we directly have the demand rate for each of the I patient types as model primitives.

As touched upon in Section 1, the distinction between the non-sequential and sequential patient-scheduler interactions reflects the differences present in various real-life appointment scheduling systems. The non-sequential model is best suited for web-based appointment scheduling systems such as www.zocdoc.com. In such systems the patient is presented with a list of time slots to choose from, which corresponds to a single offer set. In contrast, sequential scheduling reflects the iterative nature of, for instance, telephone-based appointment scheduling. Here the scheduler (e.g., the doctor’s assistant) may propose one or more slots initially, and may present more if these are rejected by the patient. While the sequential model may not capture the precise nature of the negotiation between patients and the scheduler in this case, it nevertheless provides a useful first-order approximation for this complex process.

The assumption on the unobservability of the patient type is unique in our work, and is present in all real-world systems that we consider. Web and mobile-based appointment scheduling systems either do not solicit any personal information or only collect a minimal amount of information such as the patient’s insurance plan before displaying slots. Many telephone-based schedulers only know some basic demographic information of the patients. Our empirical analysis (see Section 5.1) shows that such demographic information does not predict patient acceptance profiles (at least for the patient population we study). Even if these collected

data are useful in predicting patient preferences, most healthcare organizations lack resources (e.g., human, technology and software) to make such predictions and then use them in scheduling decisions. This is another important motivation why we choose to assume patient type is unknown in our models.

Our objective is to maximize the fill rate (or equivalently, fill count), thereby assuming that each patient contributes to the objective equally. We choose this objective for a few reasons. First, based on our conversation with healthcare practitioners, fill rate is a widely-used reporting metric for their operational and financial performance. The simplicity of this metric also makes it more trackable for analysis. Second, fairness often carries more weight than profitability in the vision of a healthcare delivery organization. Third, while different patients may bring different rewards (e.g., reimbursements) to the clinic, how to associate such rewards with patient types is not well understood in the literature. Thus, we choose a straightforward objective instead, without guessing a complicated reward structure lacking empirical support.

Finally, our discrete-time patient arrival model with at most one arrival per period is widely accepted and used by many operations management studies, including those on healthcare scheduling (e.g., Green et al. 2006, Gupta and Wang 2008, Wang and Fung 2015) and on revenue management (e.g., Talluri and Van Ryzin 2004, Bernstein et al. 2015). It can be used to approximate an inhomogeneous Poisson arrival process (Subramanian et al. 1999).

3 Non-sequential Offering

We first consider the non-sequential offering model, in which only one offer set S is presented to each arriving patient. Denote by $\mathbf{m} \leq \mathbf{b}$ a J -dimensional, non-negative integer vector that represents the current number of remaining slots of each type, and by \mathbf{e}_j the J -dimensional unit vector with its j th entry being 1 and all others zero. Define $\bar{S}(\mathbf{m}) := \{j = 1, \dots, J : m_j > 0\}$, the set of slot types with positive capacity, and $V_n(\mathbf{m})$ as the expected maximum number of appointment slots that can be booked from period n to period 1 with \mathbf{m} slots available at the beginning of period n . Note that we count time backwards.

Further, denote by $q_{ij}(S)$ the probability that slot type j is chosen conditional on a type- i patient arrival and an offer set $S \in \bar{S}(\mathbf{m})$. We have, for any j ,

$$q_{ij}(S) = \begin{cases} \frac{\Omega_{ij}}{\sum_{k \in S} \Omega_{ik}}, & \text{if } \sum_{k \in S} \Omega_{ik} > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Then, the probability that slot type j is chosen when offer set S is given is

$$q_j(S) = \sum_{i=1}^I \lambda_i q_{ij}(S), \quad (2)$$

and the no-booking probability is $q_0(S) = 1 - \sum_{j=1}^J q_j(S)$. The optimality equation is

$$V_n(\mathbf{m}) = \max_{S \subseteq \bar{S}(\mathbf{m})} \left[\sum_{j \in S} q_j(S) (1 - \Delta_{n-1}^j(\mathbf{m})) \right] + V_{n-1}(\mathbf{m}), \quad \text{for } n = N, N-1, \dots, 1, \quad (3)$$

where $V_0(\cdot) = 0$ and $\Delta_{n-1}^j(\mathbf{m}) := V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m} - \mathbf{e}_j)$ denotes the marginal benefit due to the m_j th unit of slot type j at period $n-1$.

The non-sequential offering scheme models the emerging, web and mobile-based online appointment booking systems². The most natural scheduling practice in these real-world systems is the *greedy policy*, i.e., offering all available slot types throughout the booking period. We first discuss the performance of this policy.

²The non-sequential offering model can be also appropriate for many non-medical service reservation systems, such as car service and restaurant reservations. We do not further explore these settings.

3.1 Performance of the greedy policy

Let us represent by π_0 the greedy policy, so the offer set under π_0 at state \mathbf{m} is simply $\bar{S}(\mathbf{m})$, irrespective of the period n . We denote by $V_{n,\pi_0}(\mathbf{m})$ the expected fill count attained by applying the greedy policy π_0 throughout.

To get a better sense of the performance of the greedy policy, we first provide a constant performance guarantee that states that for any set of parameters, greedy policy achieves at least half of the optimal fill count.

Theorem 1. *For any Ω , n , and \mathbf{m} , $V_n(\mathbf{m}) \leq 2V_{n,\pi_0}(\mathbf{m})$.*

It is worth noting that Theorem 1 in fact holds more broadly for all so-called *myopic policies*, which at each period offer a set maximizing the expected number of filled slots for that period. Myopic policies, however, do not have to be greedy. Performance guarantee results on myopic policies exist in various dynamic optimization settings; see, e.g., Mehta (2013), Chan and Farias (2009).

The performance bound presented in Theorem 1, while holding for any set of parameters, is relatively loose. In reality, it appears that greedy policy performs much better than this lower bound. To illustrate this, let us look at an asymptotic result, where we show that if we let the number of slot types J grow large, and scale the other parameters appropriately, the greedy policy is asymptotically optimal. In particular, we assume I , N and \mathbf{b} grow at the same speed as J , the number of slot types accepted by each patient type is bounded, and the arrival rate for each patient type decreases inversely proportional to J . See (15)-(23) in the appendix for a detailed description of the scaling regime.

Theorem 2. *If conditions (15)-(23) hold, then*

$$\lim_{J \rightarrow \infty} \frac{V_N(\mathbf{b}) - V_{N,\pi_0}(\mathbf{b})}{V_{N,\pi_0}(\mathbf{b})} = 0.$$

Naturally, the performance of any policy depends on the parameters of the system, in particular the Ω matrix which represents the patient preferences. To evaluate the performance of the greedy policy for finite regimes, we next turn towards four specific model instances of Ω matrices.

3.2 Results for specific model instances

When there are $J = 2$ slot types, the choice matrix Ω has two possible non-trivial values:

$$\Omega = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These we refer to as the N model instance (see Figure 1(a)), and the W model instance (see Figure 1(b)), respectively. These two model instances are for instance applicable to the popular Chinese scheduling system www.guahao.com.cn, which allows patients to book either a morning or an afternoon appointment for a certain day without providing more granular time interval options. In both cases we can show that it is optimal to offer all possible slots, as not doing so would unnecessarily risk sending away certain patients. This is formalized in the following result.

Proposition 1. *For the N and W model instances, the greedy policy is optimal.*

When there are $J = 3$ slot types, the simplest nontrivial choice matrix is the M model instance in Figure 1(c):

$$\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

It turns out that in this case, the greedy policy is not always³ optimal; rather, rationing of the versatile type-2 slot is needed. We define policy π_1 according to its offer set:

$$S^{\pi_1}(\mathbf{m}) := \begin{cases} \{1, 3\} & \text{if } m_1 > 0 \text{ and } m_3 > 0, \\ \bar{S}(\mathbf{m}) & \text{otherwise.} \end{cases} \quad (4)$$

³When all m_j 's are very large compared to N , the greedy policy is clearly optimal. This is, however, typically not true in the healthcare setting, given the shortage of healthcare service capacity.

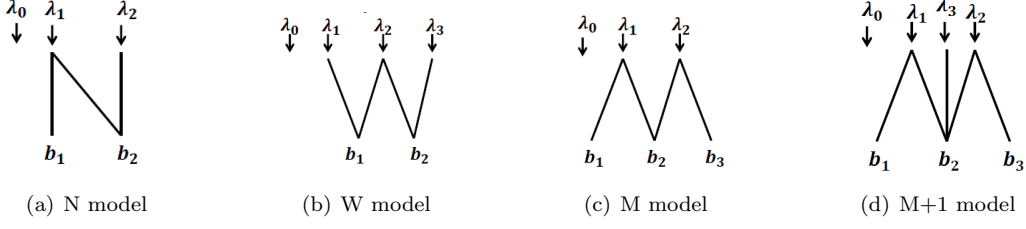


Figure 1: The N, W, M, and M+1 model instances.

So policy π_1 proposes to hold back on offering type-2 slots until either type-1 or type-3 slots are used up. We now formalize one cannot do better than this.

Proposition 2. *For the M model instance, π_1 is optimal.*

The intuition behind Proposition 2 is that blocking slot type 2 does not lead to any immediate loss of patient demand compared to offering it, while forcing early patients into *less popular* slot types (types 1 and 3). This helps to preserve the *versatile* slots (type 2) for later arrivals, when slots run low.

Following from Proposition 2, we can also show that a versatile type 2 slot is at least as valuable as one of the other two less popular slot types.

Corollary 1. *In the M model instance, for either $j = 1$ or 3 or both,*

$$V_n(\mathbf{m} - \mathbf{e}_2) \leq V_n(\mathbf{m} - \mathbf{e}_j), \quad \forall \mathbf{m} > 0, \quad n \in \{1, \dots, N\}. \quad (5)$$

Note that one of the two less popular slot types (1 and 3), however, may be strictly more valuable than the popular type 2. For example, for $\lambda_1 = \lambda_2 = 0.5$, it is easy to verify that $V_2(2, 1, 0) = 1.625 < 1.75 = V_2(2, 0, 1)$. The reason here is the following. With $m_1 = 2$ and $n = 2$, sufficient capacity is available for potential type 1 patient demand (i.e., at most 2 units). If $(m_2, m_3) = (1, 0)$, the one unit of type 2 slot has a positive probability of being taken by a type 1 patient (which would be a waste); in contrast, if $(m_2, m_3) = (0, 1)$, the one unit of type 3 slot can only be exclusively offered to type 2 patients (for whom no sufficient capacity is available), and thus this is more efficient. This simple example shows that because of patients' ability to (randomly) choose from their offer set, *less popular slots may be more valuable than versatile slots due to resource imbalance*. We shall revisit the impact of resource imbalance on the optimal offer set when discussing the next model instance.

From the above reasoning it is clear that greedy is not always optimal in the M model instance. To verify this, and to study the optimality gap, we run extensive numerical experiments, see Table 1. Here we use backward induction to determine the expected performance of the optimal policy π_1 , and compare it through simulation to that of the greedy policy π_0 . To this end we simulate replicas of 1000 days. The performance metric of interest is the relative improvement $(u_o - u_g)/u_g \times 100\%$, where u_o is the expected fill count of the optimal policy, and u_g is the average fill count over 1000 simulated days under the greedy policy. We present the relative improvement statistics in a number of scenarios, where the maximum, average and medians are taken over all possible initial capacity levels $(b_1, b_2, b_3) \in \mathbb{Z}_+^3$ such that $b_j \geq 0.2N$, $\forall j$ and $b_1 + b_2 + b_3 = N$. This corresponds to the number of scenarios in the second column. Observe that the improvement of the simple static rationing policy π_1 over the greedy policy is about 3-6%, and that it is quite robust to the changes in various model parameters. It should be noted that a 3-6% improvement is significant in the low-margin healthcare setting.

The optimality of π_1 independent of all system parameters suggests that “popularity” or “versatility” of a slot type in this case can be viewed simply in terms of the number of accepting patient types irrespective of the arrival probabilities. However, we see that this breaks down when adding another patient type to the M model instance.

The next model instance that we focus on is the M+1 model shown in Figure 1(d), with choice matrix

$$\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Table 1: Relative improvement on the fill count in the M model instance (optimal vs. greedy).

N	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	45	4.6%	3.7%	3.8%	4.3%	3.4%	3.5%	3.7%	3.0%	3.1%
30	91	5.0%	3.8%	3.9%	4.7%	3.6%	3.6%	4.0%	3.2%	3.3%
40	153	5.3%	3.9%	3.9%	4.9%	3.7%	3.7%	4.2%	3.4%	3.4%
50	231	5.6%	3.9%	3.9%	5.1%	3.8%	3.8%	4.3%	3.4%	3.5%

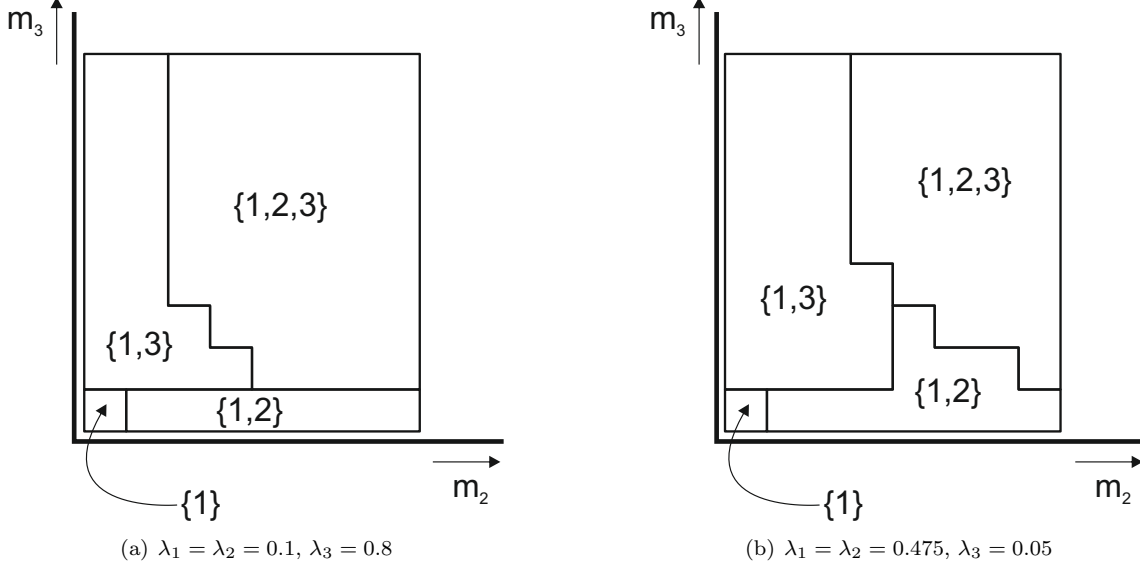


Figure 2: The optimal policy for an M+1 model instance, with $m_1 = 4$, $n = 5$.

It turns out that the elegant form of the optimal policies in the previous three cases does not carry over to the M+1 model instance. These findings, as we shall discuss in more detail below, suggest that the optimal policy for the general model can be highly complex.

To illustrate the complexity of the M+1 model instance, consider the case with $m_1 = 4$ and $n = 5$. Figure 2 shows the *unique* optimal offer set, identified with $S \subset \{1, 2, 3\}$, as a function of m_2 and m_3 . (For instance, if $S = \{1, 3\}$, it means offering slot types 1 and 3 but not slot type 2.) Consider $\lambda_1 = \lambda_2 = 0.1$, $\lambda_3 = 0.8$ (Figure 2(a)) or $\lambda_1 = \lambda_2 = 0.475$, $\lambda_3 = 0.05$ (Figure 2(b)). We see that the arrival rate now has a strong impact on the structure of the optimal policy, in contrast to the other cases we discussed so far: when λ_3 is large it is often optimal to include type-2 slots in the offer set, while for λ_3 small this is not the case. The reasoning here is that for λ_3 small the model is very close to the M model instance, in which case we know it is optimal to save versatile type-2 slots for later in the reservation process. However, not offering type-2 slots also implies turning away all type-3 patients, which explains why this slot type should be offered when λ_3 is large.

As demonstrated by Corollary 1, resource imbalance can make a less popular slot more valuable than a more popular one, which naturally would suggest an action of saving the less popular slot by only offering the versatile slot. Indeed, in Figure 2(b), we see that action $\{1, 2\}$ can be the unique optimal action even when $\mathbf{m} > 0$. This is true because m_3 is relatively small (equal to 1 or 2 in this case), while $m_1 = 4$ is ample given $n = 5$ and the symmetric arrival rates of type 1 and type 2 patients. Blocking type 3 and offering the versatile type 2 earlier rather than later can help to resolve the resource imbalance by maximizing the total expected amount of type 2 slots taken by type 3 patients.

To further evaluate the performance of the greedy policy, and the value of knowing the exact optimal

policy, we numerically evaluate these in Table 2. The setup of this table is similar to that of Table 1. When λ_3 is small the relative improvement of the optimal policy over greedy is sizable, as the model instance is very similar to the M model. As λ_3 increases the performance of the greedy heuristic improves significantly, since greedy becomes more likely to be optimal.

So it appears that the greedy policy does very well indeed, except in a few select cases. In Section 5.1 we apply this heuristic to a realistic-size problem instance, and see that its performance there is almost optimal in the non-sequential offering setting. It is useful to note, however, that for certain small and moderate cases, solving the MDP optimally may give noticeably better system performance, as illustrated by the M model instance.

Table 2: Relative improvement on the fill count in the M+1 model instance (optimal vs. greedy).

N	# of Scenarios	$(\lambda_1, \lambda_2, \lambda_3) = (9/20, 9/20, 1/10)$			$(\lambda_1, \lambda_2, \lambda_3) = (2/5, 2/5, 1/5)$			$(\lambda_1, \lambda_2, \lambda_3) = (3/10, 3/10, 2/5)$		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	45	3.1%	2.1%	2.0%	2.0%	1.1%	1.0%	0.7%	0.3%	0.2%
30	91	3.5%	2.1%	2.0%	2.3%	1.2%	1.0%	0.8%	0.3%	0.2%
40	153	3.8%	2.2%	2.0%	2.5%	1.2%	1.0%	0.8%	0.2%	0.1%
50	231	4.0%	2.2%	2.1%	2.7%	1.2%	1.0%	0.8%	0.2%	0.1%

4 Sequential Offering

Our second scheduling paradigm allows the scheduler to offer multiple sets of slots. Upon patient arrival, the scheduler chooses a K , $1 \leq K \leq J$, and sequentially presents the patient with K mutually exclusive subsets $S_1, S_2, \dots, S_K \subseteq \tilde{S}(\mathbf{m})$. We denote this action as $\mathbf{S} := S_1 - S_2 - \dots - S_K$. Denote by $\mathcal{S}(\mathbf{m})$ the set of all possible such actions at state \mathbf{m} , and by $I_k(\mathbf{S}) := \{i : \sum_{j \in S_k} \Omega_{ij} \geq 1, i \notin \cup_{l=1}^{k-1} I_l(\mathbf{S})\}$, $k = 1, \dots, K$, the set of patient types who do not accept any slot from the first $(k-1)$ offer sets but encounter at least one acceptable slot in S_k . The probability that slot type j is chosen under action \mathbf{S} may then be written as $q_j(\mathbf{S}) := \sum_{k=1}^K \sum_{i \in I_k(\mathbf{S})} \lambda_i q_{ij}(S_k)$, with $q_{ij}(\cdot)$ as in (1).

For ease of presentation, we still use $V_n(\mathbf{m})$ to denote the expected maximum number of slots that can be booked with \mathbf{m} slots available and n periods to go in this section. For an action \mathbf{S} , we let $\bigcup \mathbf{S} := \bigcup_{i=1}^K S_i$ denote the set of all slot types offered throughout action \mathbf{S} . Then, for the sequential offering model, we have

$$V_n(\mathbf{m}) = \max_{\mathbf{S} \in \mathcal{S}(\mathbf{m})} \left[\sum_{j \in \bigcup \mathbf{S}} q_j(\mathbf{S}) (1 - \Delta_{n-1}^j(\mathbf{m})) \right] + V_{n-1}(\mathbf{m}), \quad \text{for } n = N, N-1, \dots, 1, \quad (6)$$

where $V_0(\cdot) = 0$ and $\Delta_{n-1}^j(\mathbf{m}) := V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m} - \mathbf{e}_j)$ denotes the marginal benefit due to the m_j th unit of slot type j at period $n-1$. We observe that both the transition probability $q_j(\mathbf{S})$ and the set of feasible actions $\mathcal{S}(\mathbf{m})$ are much more complicated than their counterparts in the non-sequential model.

The sequential offering setting can be viewed as a generalization of non-sequential scheduling to any number $K \geq 1$ of offer sets. Consequently, it stands to reason that the greedy policy will not perform well in the sequential setting, as this would limit the scheduler to a single offer set ($K = 1$). We indeed numerically confirm this conjecture in Section 4.5. Note that, in contrast to the non-sequential case, a greedy policy is unlikely to be used in a practical setting such as telephone scheduling (because it would take too much time for the scheduler to go over every possible appointment option with the patient). To get a better intuition for the sequential case, we first consider a general setting and derive various structural results that provide more insight. We then turn to the same model instances explored in Section 3. This provides us with enough information to devise a straightforward and accurate heuristic.

4.1 Results for the general sequential offering model

We now present some properties of the sequential model with general choice matrices. First, we derive some structural properties of the value function.

Lemma 1. *The value function $V_n(\mathbf{m})$ satisfies:*

- (i) $0 \leq V_{n+1}(\mathbf{m}) - V_n(\mathbf{m}) \leq 1, \forall \mathbf{m} \geq 0, \forall n = 1, 2, \dots, N-1$; and
- (ii) $0 \leq V_n(\mathbf{m} + \mathbf{e}_j) - V_n(\mathbf{m}) \leq 1, \forall \mathbf{m} \geq 0, \forall n = 1, 2, \dots, N$.

Part (ii) of Lemma 1 implies that $V_n(\mathbf{m} + \mathbf{e}_j) \leq V_n(\mathbf{m}) + 1$, i.e., it is better to have a slot booked now rather than saving it for future. Therefore, in the context of sequential offering, it is better to keep offering slots if none has been taken so far. This is formalized in the following result, which shows that there exists an optimal sequential offering policy that exhausts all available slot types in each period.

Lemma 2. *For any Ω , \mathbf{m} , and n , there exists an optimal action \mathbf{S}^* such that $\bigcup \mathbf{S}^* = \bar{S}(\mathbf{m})$.*

Building upon Lemma 2, we are able to characterize the structure of an optimal sequential offering policy, described in the theorem below.

Theorem 3. *Let $\mathbf{m} > 0$ be the system state at period $n \geq 1$, and let j_1, j_2, \dots, j_J be a permutation of $1, 2, \dots, J$ such that $V_{n-1}(\mathbf{m} - \mathbf{e}_{j_k}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_{j_{k+1}})$, $k = 1, 2, \dots, J-1$. Then the action $\{j_1\} - \dots - \{j_J\}$ is optimal.*

Theorem 3 implies that there exists an optimal policy that offers one slot type at a time. More importantly, this result shows a specific optimal offer sequence based on the value function to go. To understand this, recall that $V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m} - \mathbf{e}_j)$ can be viewed as the value of keeping the m_j th type j slot from period $n-1$ onwards. As all patients bring in the same amount of reward, it benefits the system the most if an arrival patient can be booked for the slot type with the least value to keep, i.e., the slot type with the largest $V_{n-1}(\mathbf{m} - \mathbf{e}_j)$.

Even if the scheduler does not know the exact patient type, following the optimal offer sequence described in Theorem 3 ensures that the arriving patient takes the “least valuable” slot (as long as there is at least one acceptable slot remaining). Indeed, matching patients with slots in this way would be the best choice for the scheduler, even if she had perfect information about patient type, i.e., she knew exactly the patient type upon arrival. Following this rationale, our next result shows an interesting correspondence between (i) the sequential offering without patient type information and (ii) the non-sequential offering with *perfect* patient type information. To distinguish these two settings, we let $V_n^s(\mathbf{m})$ and $V_n^f(\mathbf{m})$ represent the value functions for settings (i) and (ii), respectively, in the next theorem.

Theorem 4. $V_n^s(\mathbf{m}) = V_n^f(\mathbf{m}), \forall \mathbf{m} \geq 0, n = 0, 1, 2, \dots, N$.

Theorem 4 points to the equivalence between the value functions in these two settings; however, it does *not* imply that the optimal sequential offering can fully elicit customer type. Specifically, optimal sequential offering happens to result in the same system state changes as if the scheduler had full information about patient type, but does not let the scheduler know exactly the patient type (see Remark 2 in Section 4.3). Nevertheless, Theorem 4 suggests that sequential offering is a useful operational mechanism to improve the organization’s efficiency in the absence of customer type information. Our numerical experiments in Section 4.5 confirm and quantify such efficiency gains.

4.2 Optimal sequential offering policies

In this section we fully characterize the optimal sequential offering policy for a large class of choice matrix instances, which include the N, M and M+1 model instances (see Figure 1). To this end, let $I(j)$ be the set of patient types who accept slot type j , i.e., $I(j) = \{i = 1, 2, \dots, I : \Omega_{ij} = 1\}, \forall j = 1, 2, \dots, J$. It makes intuitive sense that if $I(j_1) \subset I(j_2)$, then slot type j_2 is more valuable than j_1 , and thus slot type j_1 should be offered first. Combining this observation with Theorem 3 could then help us to design an optimal policy. Let us first introduce a specific class of model instances.

Definition 1. *We say that a model instance characterized by Ω is nested if for all $j_1, j_2 = 1, 2, \dots, J$ and $j_1 \neq j_2$, one of the following three conditions holds: (i) $I(j_1) \cap I(j_2) = \emptyset$, (ii) $I(j_1) \subset I(j_2)$, or (iii) $I(j_1) \supset I(j_2)$.*

Note that not all model instances are nested. One simple example is the W model instance from Figure 1(b), where $I(1) = \{1, 2\}$ and $I(2) = \{2, 3\}$. None of the conditions (i)-(iii) from Definition 1 hold in this case for $j_1 = 1$ and $j_2 = 2$. However, it is readily verified that the N, M and M+1 model instances are all nested.

Remark 1. *The concept of a nested model instance is related to the star structure considered in the previous literature on flexibility design; see, e.g., Akçay et al. (2010). Consider a system with a certain number of resource types (corresponding to slot types in our context), which can be used to do jobs of certain types (patient types in our context). A star flexibility structure is one such that there are specialized resource types, one for each job type, plus a versatile resource type that can perform all job types. The nested structure generalizes the star structure.*

It turns out that we can fully characterize an optimal policy for nested model instances as follows.

Theorem 5. *Suppose Ω is nested, any policy that offers slot type j_1 before offering slot type j_2 for any j_1, j_2 such that $I(j_1) \subset I(j_2)$ is optimal.*

Theorem 5 proposes to offer nested slot types in an increasing order of the patient types they cover. Note that when two slot types are mutually exclusive (i.e., $I(j_1) \cap I(j_2) = \emptyset$), the order in which they are offered is irrelevant, since patients that would select a slot from one set could never from the other. Using Theorem 5 we can fully characterize the optimal policy of the N, M and M+1 model instances, as formalized in the corollaries below.

Corollary 2. *For the N model instance and any n and \mathbf{m} , an optimal sequential offering policy is to offer $\mathbf{S} = \{1\} - \{2\}$.*

Corollary 3. *For the M and M+1 model instances and any n , an optimal sequential offering policy is to offer*

$$\mathbf{S} = \begin{cases} \{1, 3\} - \{2\}, & \text{if } m_1, m_2, m_3 \geq 1, \\ \{1\} - \{2\}, & \text{if } m_3 = 0, \\ \{3\} - \{2\}, & \text{if } m_1 = 0. \end{cases}$$

4.3 Beyond nested model instances

While Theorem 5 solves a large class of the sequential model instances, not all instances have a nested structure. In this section, we analyze the W model instance (see Figure 1) to glean some insights into the instances which are not nested.

To analyze the W model instance, one can formulate an MDP with three possible actions: $\{1, 2\}$, $\{1\} - \{2\}$, and $\{2\} - \{1\}$ (and the corresponding actions at the boundaries). However, there exist no straightforward offering orders for slot types, and it turns out the optimal sequential policy is state dependent. Specifically, we find that the optimal policy is a *switching curve* policy: with the availability of one type of slots held fixed, it is optimal to offer the other type of slots first as long as there is a sufficiently large amount of such slots left.

Figure 3 illustrates the optimal actions for the W model instance at different system states with $\lambda = (0.2, 0.5, 0, 3)$ and $n = 6$. The symbols “0”, “1”, “2”, “12” and “21” correspond to the actions of offering nothing, offering type 1 slots only, offering type 2 slots only, offering type 1 slots and then type 2 slots, and offering type 2 slots and then type 1 slots, respectively. The optimal actions at boundary are obvious. In the interior region of the system states, we can clearly see the switching curve structure. For instance, when the system state is (3, 3), it is optimal to offer $\{1\} - \{2\}$. When the number of type 2 slots increase to 4, then it is optimal to offer $\{2\} - \{1\}$.

The intuition behind this is different from that of the model instances considered above where patient preferences are nested (e.g., the N, M and M+1 model instances). In the W model instance, type 1 (3 resp.) patients only accept type 1 (2 resp.) slots; but type 2 patients accept both types of slots. If there are relatively more type 1 slots than type 2 slots, then it makes more sense to “divert” type 2 patients to choose type 1 slots, thus saving type 2 slots only for type 3 patients. Accordingly, the switching curve policy stipulates that type 1 slots to be offered first, ensuring that type 2 patients if any will pick type 1 slots. The intuition above is formalized in the proposition below.

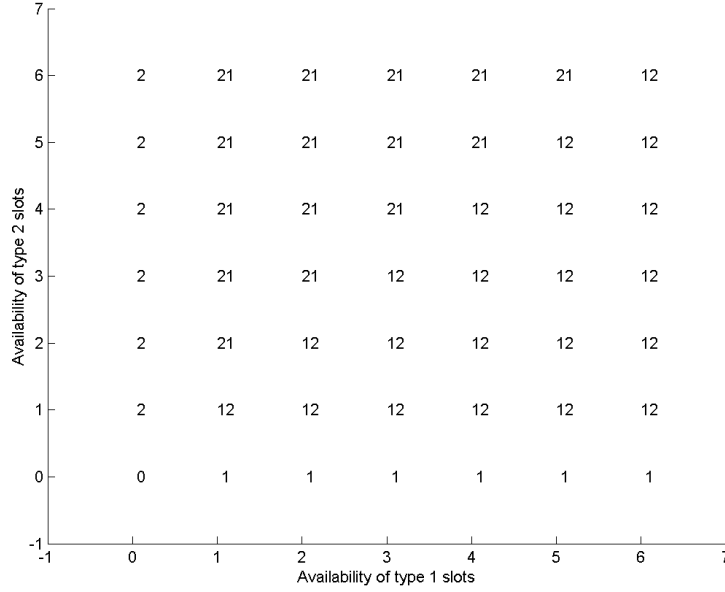


Figure 3: Structure of the optimal policy under W model instance with sequential offers.

Proposition 3. Consider the W model instance with sequential offers. Given m_2 , if there exists an m_1^* such that the optimal action at state (m_1^*, m_2) is $\{1\} - \{2\}$, then $\forall \mathbf{m} \in \{(m_1, m_2), m_1 \geq m_1^*\}$, the optimal action is $\{1\} - \{2\}$. Similarly, given m_1 , if there exists an m_2^* such that the optimal action at state (m_1, m_2^*) is $\{2\} - \{1\}$, then $\forall \mathbf{m} \in \{(m_1, m_2), m_2 \geq m_2^*\}$, the optimal action is $\{2\} - \{1\}$.

Remark 2. In Section 4.2, we state that sequential offering may not fully reveal exact patient types, but allows the system to evolve in the same optimal way as if the scheduler knew exactly the patient type. We use the W model instance to illustrate this point. Consider the W model instance with non-sequential offers and the scheduler knows the exact type of arriving patients. Suppose the optimal action is to offer $\{1\}$ when type 1 or type 2 patients arrive; and to offer $\{2\}$ when type 3 patient arrives. Now, in a sequential offering model where the scheduler does not know the exact type of arriving patients, the scheduler would have offered $\{1\} - \{2\}$ to any arriving patient. If we encountered type 1 or 2 patients, type 1 slot would be taken, but we do not know the exact type of this patient (we know she must be either type 1 or type 2 though); if type 3 patient arrived, she would reject type 1 slot, but take type 2 slot. In this way, the system evolves as if the scheduler had perfect information on patient type.

The structural properties of the optimal policy described in Proposition 3 are likely the best we can obtain for the W model instance; the exact form of the optimal policy depends on model parameters and the system state, much like with the M+1 model instance in the non-sequential case. If patient preference structures become more complicated, it is very difficult, if not impossible, to develop structural properties for the optimal sequential offering policy. Thus, for model instances that do not satisfy the conditions of Theorem 5, we propose an effective heuristic below.

4.4 The “Drain” Heuristics

If patient preferences are not nested, the analysis of the W model instance suggests that the optimal policy is to offer slots with more capacity relative to its patient demand. Inspired by this observation and using the idea of fluid models, we propose the following heuristic algorithm which aims to “drain” the abundant slot type first followed by less abundant ones. This heuristic aims to have all slot types emptied simultaneously, thus maximizing the fill rate. Specifically, the drain algorithm works in the following simple way. At period

n and for each slot type $j \in \bar{S}(\mathbf{m})$, we calculate

$$I_j := \frac{m_j}{n \sum_{i=1}^I \lambda_i \frac{\Omega_{ij}}{\sum_{k \in \bar{S}(\mathbf{m})} \Omega_{ik}}}. \quad (7)$$

Note that $\frac{\Omega_{ij}}{\sum_{k=1}^J \Omega_{ik}}$ represents the share of type i patients who will choose type j slots, assuming all available slot types are offered simultaneously. Taking expectation with respect to the patient type distribution and multiplying by n , the number of patients to come, the denominator of (7) can be viewed as the expected load on type j slots in the next n periods. As a result, the index I_j can be regarded as the ratio between capacity left and “expected” load.

The drain algorithm is then to calculate all I_j s at the beginning of each period, and to offer slots in decreasing order of the I_j . The algorithm calls for offering slot types with larger I_j first, as these slot types have relatively more capacity compared to demand. In other words, a slot type with a larger I_j is likely to have a smaller marginal value to keep, and thus can be offered earlier. We will test the performance of this algorithm in Section 4.6.

4.5 Value of Sequential Offering

In the next two sections we present some numerical results on the sequential model. In this section we investigate the value of sequential scheduling by comparing the optimal sequential policy to the greedy policy (which is essentially non-sequential); in Section 4.6 we study the performance of the drain heuristic.

In this section we focus on the N , W and M model instances. To provide a robust performance evaluation, we vary the mix of patient types, the number of periods remaining and the initial capacity vectors. As with Tables 1 and 2 we look at the relative improvement.

Table 3 presents the maximum, average and median percentage improvement in fill count by following an optimal sequential offering policy compared to the greedy policy in the N model instance; and these improvement statistics are taken over all initial capacity vectors (b_1, b_2) such that $(b_1, b_2) \in \{(x, y) \in \mathbb{Z}_+^2 : x, y \geq 0.2N, x + y = N\}$. We also vary (λ_1, λ_2) as shown in the first row of the table. Tables 4 and 5 present the similar information for the W and M model instances, respectively. For the M model instance, we consider all the initial capacity vectors $(b_1, b_2, b_3) \in \{(x, y, z) \in \mathbb{Z}_+^3 : x, y, z \geq 0.2N, x + y + z = N\}$.

Table 3: Fill Count Improvement under the N Model instance (Optimal Sequential vs. Greedy).

N	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$			$(\lambda_1, \lambda_2) = (3/4, 1/4)$		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	13	16.0%	10.6%	12.4%	10.8%	9.0%	9.6%	13.2%	6.2%	5.5%
30	19	16.8%	10.9%	12.6%	11.1%	9.3%	9.6%	14.0%	6.3%	5.4%
40	25	17.2%	11.1%	12.5%	11.2%	9.5%	9.9%	14.5%	6.4%	5.3%
50	31	17.5%	11.2%	12.9%	11.3%	9.6%	9.8%	14.8%	6.4%	5.3%

Table 4: Fill Count Improvement under the W Model instance (Optimal Sequential vs. Greedy).

N	# of Scenarios	$(\lambda_1, \lambda_2, \lambda_3) = (1/3, 1/3, 1/3)$			$(\lambda_1, \lambda_2, \lambda_3) = (1/5, 1/2, 3/10)$			$(\lambda_1, \lambda_2, \lambda_3) = (1/10, 3/10, 3/5)$		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	13	8.2%	6.1%	6.5%	10.8%	6.6%	7.0%	11.2%	7.7%	8.3%
30	19	9.0%	6.6%	7.9%	11.8%	7.0%	7.2%	11.6%	8.1%	9.2%
40	25	9.5%	6.9%	7.9%	12.3%	7.2%	7.3%	12.0%	8.3%	9.4%
50	31	9.8%	7.1%	8.3%	12.7%	7.3%	7.3%	12.2%	8.4%	9.4%

We observe that the efficiency gains in the M and W model instances are robust to the initial patient type mix. The efficiency gain in the W model instance is about 6-7% on average, and can be as high as 13%. The efficiency gain in the M model instance is slightly higher. For the N model instance, the gain is

Table 5: Fill Count Improvement under the M Model instance (Optimal Sequential vs. Greedy).

N	# of Scenarios	$(\lambda_1, \lambda_2) = (1/2, 1/2)$			$(\lambda_1, \lambda_2) = (1/3, 2/3)$			$(\lambda_1, \lambda_2) = (1/4, 3/4)$		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	45	11.9%	7.8%	7.5%	12.2%	7.8%	7.4%	10.4%	7.2%	7.3%
30	91	13.0%	7.6%	7.3%	13.0%	7.9%	7.5%	10.9%	7.6%	7.6%
40	153	13.6%	7.5%	6.9%	13.5%	8.0%	7.8%	11.0%	7.8%	7.9%
50	231	14.1%	7.4%	6.9%	13.7%	8.1%	7.9%	11.2%	8.0%	8.2%

relatively more sensitive to patient type mix, and ranges between 6-11% on average. In certain cases, the efficiency gain in the N model instance can be as high as 18%. These numerical findings show that sequential offering holds strong potentials to improve the operational efficiency in appointment scheduling systems.

4.6 Performance of the “Drain” Heuristics

We compare our “drain” heuristics developed in Section 4.4 with three benchmark policies in the sequential offering setting.

- The *optimal sequential offering policy*.
- The *random sequential offering policy* which offers available slot types one at a time in a permutation chosen uniformly at random. This policy mimics the existing practice of telephone scheduling, which is often done without careful planning.
- The *greedy policy* which offers all available slot types at once. This is perhaps the most commonly-used non-sequential offering policy, and resembles the current practice of online scheduling.

We focus on the M and W model instances in this section, and use the combinations of parameters as in earlier sections. The performance of the optimal sequential offering policy is evaluated by backward induction. The performances of the other three policies including the drain heuristic are evaluated by running a discrete event simulation with 1000 days replication and then computing the average fill count per day. We present the percentage optimality gap of drain, and the relative improvement of drain over the random sequential offering policy and the greedy policy. Detailed results are shown in Tables 6 and 7.

Table 6: Comparison of the Drain Heuristic with Other Scheduling Policies (The M Model instance).

(λ_1, λ_2)	N	# of Scenarios	% Optimality Gap			% Imp. over Random Sequential			% Imp. over Greedy		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
$(1/2, 1/2)$	20	45	-1.4%	-0.7%	-0.8%	12.3%	7.0%	6.9%	12.1%	7.0%	6.8%
	30	91	-0.9%	-0.6%	-0.6%	13.5%	7.1%	6.7%	13.7%	7.1%	6.5%
	40	153	-0.7%	-0.5%	-0.5%	13.8%	7.0%	6.4%	14.0%	7.0%	6.3%
	50	231	-0.6%	-0.4%	-0.5%	14.2%	7.0%	6.4%	13.9%	7.0%	6.4%
$(1/3, 2/3)$	20	45	-1.1%	-0.4%	-0.4%	13.2%	7.4%	6.8%	11.8%	7.4%	7.1%
	30	91	-0.8%	-0.3%	-0.3%	13.7%	7.6%	7.1%	13.4%	7.6%	7.1%
	40	153	-0.7%	-0.3%	-0.3%	13.7%	7.7%	7.4%	13.8%	7.7%	7.7%
	50	231	-0.6%	-0.2%	-0.3%	14.2%	7.9%	7.7%	14.0%	7.8%	7.9%
$(1/4, 3/4)$	20	45	-0.9%	-0.2%	-0.2%	11.1%	6.9%	6.7%	10.8%	6.9%	6.9%
	30	91	-0.7%	-0.2%	-0.2%	11.0%	7.4%	7.4%	11.3%	7.4%	7.4%
	40	153	-0.9%	-0.2%	-0.1%	11.4%	7.7%	7.9%	11.5%	7.6%	7.8%
	50	231	-0.6%	-0.1%	-0.2%	11.8%	7.8%	8.0%	11.6%	7.9%	8.0%

For the M model instance, the optimality gap of drain is on average within 0.7% (max 1.4%) across all 1560 scenarios we tested. We see an average 7-8% improvement (max 14%) if using drain compared to using

Table 7: Comparison of the Drain Heuristic with Other Scheduling Policies (The W Model instance).

$(\lambda_1, \lambda_2, \lambda_3)$	N	# of Scenarios	% Optimality Gap			% Imp. over Random Sequential			% Imp. over Greedy		
			Max	Average	Median	Max	Average	Median	Max	Average	Median
$(1/3, 1/3, 1/3)$	20	13	-0.2%	0.0%	0.1%	8.5%	6.1%	6.8%	8.0%	6.1%	6.9%
	30	19	-0.7%	0.0%	0.0%	9.6%	6.5%	7.9%	9.1%	6.5%	7.6%
	40	25	-0.2%	0.0%	0.0%	9.8%	7.0%	8.1%	9.7%	6.9%	7.8%
	50	31	-0.2%	0.0%	0.0%	10.2%	7.1%	7.9%	10.3%	7.0%	7.9%
$(1/5, 1/2, 3/10)$	20	13	-0.1%	0.1%	0.1%	10.9%	6.7%	6.9%	10.8%	6.6%	6.7%
	30	19	-0.4%	0%	0.0%	11.9%	6.9%	7.3%	11.7%	6.9%	6.9%
	40	25	-0.2%	0.0%	0.0%	12.5%	7.2%	7.5%	12.5%	7.2%	7.2%
	50	31	-0.2%	0.0%	0.0%	12.8%	7.3%	7.4%	12.6%	7.3%	7.3%
$(1/10, 3/10, 3/5)$	20	13	-0.7%	-0.2%	-0.1%	10.9%	7.5%	8.1%	11.6%	7.8%	8.5%
	30	19	-0.6%	-0.1%	-0.1%	11.5%	8.0%	9.3%	11.9%	8.1%	9.1%
	40	25	-0.5%	0.1%	0.0%	12.3%	8.4%	9.4%	12.3%	8.4%	9.5%
	50	31	-0.4%	-0.1%	-0.1%	12.2%	8.3%	9.4%	12.4%	8.4%	9.3%

random sequential or greedy. In the W model instance, the optimality gap of drain is on average within 0.2% (max 0.7%) in all 264 scenarios we tested. On average, drain makes a 6-8% improvement over random sequential or greedy with the maximum improvement up to 13%. It is worth remarking upon that in both cases the random sequential heuristic has about the same performance as greedy. So although the former is a sequential policy and the latter is not, the potential of the former is not exploited due to the careless choice of the offered slots.

5 Case Study

We now present the results of a survey study to further motivate the choice model introduced in Section 2. We then apply the heuristics proposed in Sections 3 and 4 to the outcome of this survey to obtain realistic estimates of the performance of these heuristics.

5.1 Survey Study

Previous research has shown that patients value more flexibility in choice of time in a day when seeking care from their doctors; see, e.g., Rubin et al. (2006). No existing work, however, has explicitly studied how patients would choose among multiple possible time slots in a day. To gain some understanding of patient preference in time of day when scheduling appointments, we carried out an online survey study. We recruited participants on Amazon Mechanical Turk (MTurk), which is an online platform with an integrated participant compensation system, a large, diverse, reliable participant pool, and a streamlined process for participant recruitment and data collection (Paolacci et al. 2010).

The survey questionnaire consisted of two parts. In the first portion, participants were asked to imagine that they need to book a non-urgent appointment in the near future. Then they were asked to indicate their preferred time of day to visit their primary care doctors, taking their current work/life schedule into account. Specifically, participants were asked to indicate *all* time slots listed below acceptable to them: Early Morning (8am-10am); Late Morning (10am-12pm); Noon (12pm-1pm); Early Afternoon (1pm-3pm); Late Afternoon (3pm-5pm); Evening (5pm-8pm); and No Preference (any time above works). Note that participants, depending on their preference, may choose multiple time slots. In the second portion of the survey, participants completed a series of demographic questions regarding their age, sex, ethnicity, insurance plans, etc.

We required that the respondents have to reside in the US to ensure that our appointment options reflected options participants might have when making medical appointments. The experiment was conducted on 271 subjects in July 2013. This online sample has an average age of 28 with range 18-63. The participants

are relatively well educated (52% with a college degree or higher), dominated by white (74.2%) and males (67.1%). A large percentage of them have insurance coverage by commercial plans (60.9%) and many are full- or part-time employed (48%). This demographic profile is consistent with other reported survey samples from the Amazon Mturk (Paolacci et al. 2010).

Our initial analysis shows that there are 37 distinct acceptable combinations of time slots as indicated by the participants. However, many of these combinations are associated with very few, say 1 or 2, participants. To conduct a more meaningful analysis, we combine some of the time slots into “wider” time windows if there is a high correlation in participants’ preferences. Following this analysis, we group time slots into three grand time windows: early day from 8am-10am (ED), mid-day from 10am to 3pm (MD) and late-day from 3pm to 8pm (LD). Participants who indicate acceptance of any sub-time slots within one of such grand time windows are considered accepting that particular grand time window. All participants indicated at least one preference, and we obtain 7 distinct combinations of these grand windows, each representing one unique participant preference profile in time of day (see Table 8). We observe that any of these preference profiles is associated with a nontrivial percentage ($\geq 7.7\%$) of participants. This clearly shows that, at least in this sample, participants exhibit strong and heterogeneous preference in times of day when booking appointments.

Table 8: Participant Preferences in Time Windows.

	Count	Percentage
ED only	23	8.5%
MD only	58	21.3%
LD only	31	11.4%
ED and MD	40	14.7%
ED and LD	21	7.7%
MD and LD	43	15.8%
ED, MD and LD	56	20.6%
Total	272	100%

Our next analysis explores if any of the patient demographic information may explain such heterogeneity. If so, providers may use these demographic variables to better predict patient choice and manage their practice. Indeed, the popular online scheduling service vendor www.zocdoc.com discussed above may ask for such information (e.g., patient insurance plans), before showing appointment options to patients. Interestingly, however, we find little evidence in this participant pool that demographic information explains patient heterogeneous preference in time of day. Specifically, we use the Chi-square test to determine if the distributions of patient preference profiles are the same across different demographic categories (our null hypothesis). We focus on variables commonly known/used by providers when scheduling appointments. We cannot reject the null hypothesis for gender ($p=0.59$), race ($p=0.20$), education level ($p=0.16$), and insurance plan used by participants ($p=0.21$). We find some evidence that age may predict patient preferences, but the statistical significance is marginal ($p=0.03$). Therefore, it is unlikely that providers can use commonly-known demographic variables to further refine predictions of patient preference, at least in the population of our study.

5.2 Comparison of Scheduling Policies

The above survey results provide a realistic setting to compare various scheduling policies. We can view ED, MD and LD as 3 distinct slot types and we have 7 patient types:

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The percentages in Table 8 delineate the distribution of patient types. We use this model to evaluate the performance of the two heuristics presented in Sections 3 and 4: greedy for the non-sequential model and the drain heuristic for the sequential model. We consider $N = 20, 30, 40, 50$, and vary initial capacity vector $(b_1, b_2, b_3) \in \{(x, y, z) \in \mathbb{Z}_+^3 : x, y, z \geq 0.2N, x + y + z = N\}$.

In Table 9 we show the relative improvement of the optimal policy for non-sequential model compared to the greedy policy. We note that in this setting the greedy policy does exceedingly well, and that the relative improvement of the optimal policy over greedy is never more than 0.3%. This is in line with the asymptotic optimality presented in Theorem 2 and the good performance of the greedy policy in certain M+1 model instances.

Table 9: Comparison of the greedy Heuristic and optimal policy (Survey data).

N	# of Scenarios	% Optimality Gap		
		Max	Average	Median
20	45	0.1%	0.0%	0.0%
30	91	0.2%	0.0%	0.0%
40	153	0.3%	0.0%	0.0%
50	231	0.3%	0.1%	0.0%

We now do a similar comparison for the drain heuristic in the sequential setting. Using the survey data, we test the performance of our drain heuristic, and compare it with three benchmark policies discussed in the last section: the optimal sequential offering policy, the random sequential offering policy and the greedy policy. In the model instance obtained from the survey patient preferences are not nested, and therefore the optimal sequential offering policy does not have the simple form described in Theorem 5. However, we can numerically compute the performance of the optimal policy.

Table 10: Comparison of the Drain Heuristic with Other Scheduling Policies (Survey data).

N	# of Scenarios	% Optimality Gap			% Imp. over Random Sequential			% Imp. over Greedy		
		Max	Average	Median	Max	Average	Median	Max	Average	Median
20	45	-0.3%	0.0%	0.0%	13.8%	8.0%	8.2%	14.6%	8.0%	8.1%
30	91	-0.4%	0.0%	0.0%	15.3%	8.4%	8.4%	15.6%	8.5%	8.3%
40	153	-0.3%	0.0%	0.0%	16.9%	8.6%	8.6%	16.1%	8.6%	8.7%
50	231	-0.3%	0.0%	0.0%	17.4%	8.8%	8.6%	17.4%	8.7%	8.5%

The results from this comparison are reported in Table 10. We observe that the drain heuristic performs extremely well in this realistic setting; the maximum percentage optimality gap only 0.4%. Compared to the random sequential offering policy, the percentage improvement of drain has a median of 8-9% and can be up to more than 17%. We see similar percentage improvement when comparing drain to the greedy policy.

6 Discussion and Concluding Remarks

Motivated by the use of appointment templates in healthcare scheduling practice, we study how to offer appointment slots to patients in order to maximize the utilization of provider time. Our work complements the existing literature on the design of appointment templates. We consider two common ways that patients may interact with a scheduler in the current healthcare market: online scheduling and traditional telephone scheduling. Each scheduling paradigm has its own unique features. The current online scheduling platform usually allows providers to offer appointment slots once, while the traditional scheduling process often goes through a negotiation process between patients and scheduler. We develop two models, non-sequential scheduling and sequential scheduling, to capture these two different types of interactions.

For the non-sequential offering model we first show that the default greedy policy is a 2-approximation and is asymptotically optimal in certain regimes. We then characterize the optimal policy for a few special

instances, and demonstrate that the optimal policy can be highly complex in general. We identify certain problem instances where the greedy policy is suboptimal, but show through analytical and numerical results that for most moderate and large instances, the greedy policy performs remarkably well. Thus, our analysis of the non-sequential model suggests that offering all available appointment options for patients on the online scheduling website is likely to be a good idea, if the website only allows offering appointment slots once (and has limited information on patient preferences/choices).

For the sequential offering model, we show that there exists an optimal policy that offers slot types one at a time based on their marginal values. We are able to determine these values for a broad class of nested model instances, which allow us to explicitly derive the optimal policy in these cases. We then present a drain heuristic to deal with the cases where we cannot solve the optimal policy in closed-form, and numerically show that it is near-optimal.

6.1 Managerial Implications

In our numerical experiments, we find that sequential offering, if done properly, can make a substantial improvement in slot utilization compared to two benchmark policies (random sequential offering policy which mimics the existing practice of telephone scheduling, and the greedy policy that resembles the current practice of online scheduling). In our case study based on realistic patient choice parameters estimated from a survey study, the median improvement of the drain heuristic over these two default policies is 8-9%, and the maximum improvement is 17%. Consider a single primary care provider who has set a 30-patient daily appointment template and works 250 days a year. An 8% increase in fill rate means that about 600 additional slots will be filled in a year. For a primary care visit, the reimbursement rate is usually \$75-\$200 depending on the payer and the complexity of the visit. Based on these estimates, the revenues gain by carefully designing the offering sequence can be around \$45k-\$120k per year for this single primary care provider. This impressive improvement has two implications. First, it suggests substantial room in improving the traditional scheduling practice by optimizing the offer sequence of appointment slots. Second, because the optimal sequential offering leads to the same fill rate as the optimal non-sequential offering when exact patient type is known (Theorem 4), this considerable revenue improvement can be viewed as the *value of information* on patient preference and choice behavior.

Another important observation from our numerical experiments is that the two benchmark policies have very similar performances (see Tables 6, 7 and 10). This implies that current online scheduling (that offers all available slots) and traditional telephone scheduling (without a careful offer sequence) would result in similar fill rates. Thus, one should *not* expect that implementing an online scheduling system in place of traditional telephone scheduling may automatically lead to more appointments booked. However, as discussed above, one may improve the performance of online scheduling by designing an interfaces that allows sequential offers, or by collecting information on patient choice behavior and then making one-time offer in a smarter way (see also the literature on customized assortment planning, e.g., Bernstein et al. 2015).

6.2 Implementation Issues

Implementing any optimization model in practice requires pre-optimization and post-optimization phases. Before implementing our models, a provider will first need to estimate the patient choice matrix, total daily demand and arrival rate for each patient type. Patient choice matrix and arrival rate for each patient type may be estimated by a simple survey (like ours), staff impression or historical patient record data. If a practice has been established for some time, daily patient demand (i.e., demand to one single day during the whole booking horizon which may consist of multiple days) is likely to be in some equilibrium. We can then treat demand to each day separately and apply our single-day scheduling model to each day in question: in the online setting, we could have day-specific scheduling policies given specific daily demand pattern; in the telephone setting, we could have the scheduler first ask which day(s) the arriving patient prefers before offering time slots in the day(s). During the implementation of our models, demand patterns may change. Providers may need to periodically collect new data to refine scheduling policies until a new equilibrium is reached. Indeed, the rationale behind these implementation ideas is in fact the motivation and justification for the prior literature that develops single-day models to inform the management of multi-day operations; see, e.g., Gupta and Wang (2008), Wang and Gupta (2011) and Green et al. (2006).

6.3 Future Directions

In summary, our work provides the first framework to model, compare and improve two main scheduling paradigms used in practice. Our study also suggests several directions for future research. First, our sequential offering model is a “stylized” model for telephone scheduling. Future research may focus on understanding and modeling the detailed negotiation process between the patients and the scheduler. Second, different patient types may have different rewards. For instance, there exists variation in reimbursement rates across different insurance plans. But as discussed earlier, the relationship between the rewards and patient types is not well understood. It would be useful to empirically investigate this relationship, and then develop scheduling models explicitly capturing heterogeneous rewards among patient types. Third, we assume a specific model for patient choice; it would be interesting to consider scheduling decisions under different choice models. For instance, patients may not break ties uniformly among multiple acceptable slot types but should do so based on some ranks, e.g., different (in)convenience levels of them. This, however, would also require more data collection in order to estimate patient choices.

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Appendix: Proofs of the Results

A Preliminarily results

We first state and prove an auxiliary lemma on the structural results of the value function for the non-sequential offering model. This lemma will be used in proving other results in the paper.

Lemma 3. *Let Ω be a preference matrix, $\mathbf{m} \geq 0$, $j = 1, \dots, J$ and $n \in \{1, \dots, N\}$, then the value function $V_n(\mathbf{m})$ satisfies*

- (i) $0 \leq V_{n+1}(\mathbf{m}) - V_n(\mathbf{m}) \leq 1$; $\forall n = 0, 1, 2, \dots$;
- (ii) $0 \leq V_n(\mathbf{m} + \mathbf{e}_j) - V_n(\mathbf{m}) \leq 1$; $\forall n = 0, 1, 2, \dots$;
- (iii) if $\lambda_0 > 0$, then $V_n(\mathbf{m} + \mathbf{e}_j) - V_n(\mathbf{m}) < 1$; $\forall n = 1, 2, \dots$.

These monotonicity results are quite intuitive. Properties (i) and (ii) state that the optimal expected reward is increasing in the number of patients and the number of slots left and the changes in the optimal expected reward are bounded by the changes in the number of patients to go and the number of slots available. Property (iii) suggests that if there is a strictly positive probability that no patients would come in each period, then the increase of the optimal expected reward is strictly smaller than that of the available slots.

Proof. Proof. We use induction to prove this lemma. We first prove the first two properties. For $n = 0$, these two properties hold trivially. Suppose that they also hold up to $n = t$. Consider $n = t + 1$. Let $\mathbf{g}_t^*(\mathbf{m})$ represent the optimal decision rule in period t when the system state is \mathbf{m} . Let $V_s^{\mathbf{f}}(\mathbf{m})$ be the expected number of slots filled given that the decision rule \mathbf{f} is taken at stage s and from stage $s - 1$ onwards the optimal decision rule is used. Let $p_k(\mathbf{m}, \mathbf{f})$ be the probability that a type k slot is booked at state \mathbf{m} if action \mathbf{f} is taken. It follows that

$$\begin{aligned} V_{t+1}(\mathbf{m}) &\geq V_{t+1}^{\mathbf{g}_t^*(\mathbf{m})}(\mathbf{m}) = \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_t^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} - \mathbf{e}_k)] \\ &\geq \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_t^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_{t-1}(\mathbf{m} - \mathbf{e}_k)] \\ &= V_t(\mathbf{m}), \end{aligned}$$

where the first inequality is due to the definition of $V_{t+1}(\mathbf{m})$ and the second inequality follows from the induction hypothesis. Following a similar argument and fixing $j \in \{1, 2, \dots, J\}$, we have

$$\begin{aligned} V_{t+1}(\mathbf{m} + \mathbf{e}_j) &\geq V_{t+1}^{\mathbf{g}_{t+1}^*(\mathbf{m})}(\mathbf{m} + \mathbf{e}_j) \\ &= \sum_{k=0}^J p_k(\mathbf{m} + \mathbf{e}_j, \mathbf{g}_{t+1}^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_k)] \\ &= \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_k)] \\ &\geq \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} - \mathbf{e}_k)] \\ &= V_{t+1}(\mathbf{m}), \end{aligned}$$

where the second equality results from the decision rules and the state transition probability (2).

To show the RHS of the inequality in (i) for $n = t + 1$, note that

$$\begin{aligned}
& V_{t+1}(\mathbf{m}) - V_t(\mathbf{m}) \\
&= \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) [\mathbb{1}_{\{k>0\}} + V_t(\mathbf{m} - \mathbf{e}_k)] - V_t(\mathbf{m}) \\
&= \sum_{k=1}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) + \sum_{k=0}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) [V_t(\mathbf{m} - \mathbf{e}_k) - V_t(\mathbf{m})] \\
&\leq \sum_{k=1}^J p_k(\mathbf{m}, \mathbf{g}_{t+1}^*(\mathbf{m})) \\
&\leq 1,
\end{aligned}$$

where the first inequality follows from that $V_t(\mathbf{m} - \mathbf{e}_k) \leq V_t(\mathbf{m})$, which has been shown above.

To show the RHS of the inequality in (ii) for $n = t + 1$, we define a decision rule \mathbf{h} in period $t + 1$ such that $\mathbf{h} = \mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j)$ except $h_j = 0$. It follows that

$$V_{t+1}(\mathbf{m} + \mathbf{e}_j) - V_{t+1}(\mathbf{m}) \leq V_{t+1}(\mathbf{m} + \mathbf{e}_j) - V_{t+1}^{\mathbf{h}}(\mathbf{m}), \quad (8)$$

because \mathbf{h} may not be the optimal given system state \mathbf{m} at period $t + 1$. For $u = 1, 2, \dots, J$, let

$$q_u = p_u(\mathbf{m} + \mathbf{e}_j, \mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j))$$

and

$$q'_u = p_u(\mathbf{m}, \mathbf{h}).$$

It is easy to check that $q_u \leq q'_u, \forall u \neq 0, j$ and $q'_j = 0$. Now, let $\Omega_i = (\Omega_{i1}, \Omega_{i2}, \dots, \Omega_{iJ})$ and use $\langle \cdot, \cdot \rangle$ to represent the inner product. We have that

$$\sum_{u=1}^J q_u = \sum_{i=1}^I \lambda_i \mathbb{1}_{\{\langle \Omega_i, \mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j) \rangle > 0\}} \geq \sum_{i=1}^I \lambda_i \mathbb{1}_{\{\langle \Omega_i, \mathbf{h} \rangle > 0\}} = \sum_{u=1}^J q'_u,$$

because $\mathbf{h} = \mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j)$ except $h_j = 0$. Therefore, $q_0 = 1 - \sum_{u=1}^J q_u \leq 1 - \sum_{u=1}^J q'_u = q'_0$. Define $\delta_u = q'_u - q_u$ for $u \neq j$. It is clear that $\delta_u \geq 0, \forall u \neq j$, and we note the following relationship.

$$q_j = 1 - \sum_{u \neq j} q_u = 1 - \sum_{u \neq j} (q'_u - \delta_u) = \sum_{u \neq j} \delta_u.$$

Now, we can continue the inequality (8) as follows.

$$\begin{aligned}
& V_{t+1}(\mathbf{m} + \mathbf{e}_j) - V_{t+1}^{\mathbf{h}}(\mathbf{m}) \\
&= \sum_{u=0}^J q_u [\mathbb{1}_{\{u>0\}} + V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_u)] - \sum_{u=0}^J q'_u [\mathbb{1}_{\{u>0\}} + V_t(\mathbf{m} - \mathbf{e}_u)] \\
&= (1 - q_0) - (1 - q'_0) + \sum_{u=0}^J q_u V_t(\mathbf{m} + \mathbf{e}_j - \mathbf{e}_u) - \sum_{u=0}^J q'_u V_t(\mathbf{m} - \mathbf{e}_u) \\
&= \delta_0 + \sum_{u \neq j} q_u (V_t[\mathbf{m} + \mathbf{e}_j - \mathbf{e}_u] - V_t(\mathbf{m} - \mathbf{e}_u)) + q_j V_t(\mathbf{m}) - \sum_{u \neq j} \delta_u V_t(\mathbf{m} - \mathbf{e}_u) \\
&= \delta_0 + \sum_{u \neq j} q_u (V_t[\mathbf{m} + \mathbf{e}_j - \mathbf{e}_u] - V_t(\mathbf{m} - \mathbf{e}_u)) + \sum_{u \neq j} \delta_u V_t(\mathbf{m}) - \sum_{u \neq j} \delta_u V_t(\mathbf{m} - \mathbf{e}_u) \\
&= \delta_0 + \sum_{u \neq j} q_u (V_t[\mathbf{m} + \mathbf{e}_j - \mathbf{e}_u] - V_t(\mathbf{m} - \mathbf{e}_u)) + \sum_{u \neq 0, j} \delta_u [V_t(\mathbf{m}) - V_t(\mathbf{m} - \mathbf{e}_u)] \\
&\leq \delta_0 + \sum_{u \neq j} q_u + \sum_{u \neq 0, j} \delta_u \\
&= 1,
\end{aligned}$$

where the last inequality comes from the induction hypothesis for property (ii).

As for property (iii), first note that it trivially holds for $n = 1$. We can then follow similar induction steps as those used to prove the RHS of the inequality in property (ii) to complete the proof. \square

B Proof of Theorem 1

Proof. Proof. We prove this by induction. Let Ω be any preference matrix. For $n = 1$, π_0 is optimal and thus $V_1(\mathbf{m}) = V_{1,\pi_0}(\mathbf{m}) \leq 2V_{1,\pi_0}(\mathbf{m})$. Suppose the desired result holds for any $n \leq k - 1$ and state \mathbf{m} .

Now we consider two systems, one under an optimal policy and the other under π_0 , both starting from state \mathbf{m} in period $n = k$ and operating independently from each other. We denote by $L_k^*(\mathbf{m})$ the slot type filled in period k in the first system (i.e., using an optimal policy), and by $L_k^{\pi_0}(\mathbf{m})$ the slot type filled in period k in the second system (i.e., using π_0). These two random variables are independent and we shall next condition on them. Specifically, let $V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m}))$ denote the value attained in the first system conditioning on these two random variables. Let $\mathcal{J} = \{1, 2, \dots, J\}$ be the set of patient types. We have that

$$\begin{aligned} V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) &= E[\mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} | L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})] + V_{k-1}(\mathbf{m} - \mathbf{e}_{L_k^*(\mathbf{m})}) \\ &= \mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} + V_{k-1}(\mathbf{m} - \mathbf{e}_{L_k^*(\mathbf{m})}) \\ &\leq \mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} + V_{k-1}(\mathbf{m}) \end{aligned} \quad (9)$$

$$\leq \mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} + \mathbb{1}_{\{l > 0\}} + V_{k-1}(\mathbf{m} - \mathbf{e}_l), \quad \forall l \in \bar{S}(\mathbf{m}) \cup \{0\}, \quad (10)$$

where inequality (9) follows from the left inequality of Lemma 3 (ii) and inequality (10) holds due to the right inequality of Lemma 3 (ii). We now let $l = L_k^{\pi_0}(\mathbf{m})$ in (10) and in turn have

$$V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) \leq \mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} + \mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m}) > 0\}} + V_{k-1}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})}). \quad (11)$$

Further applying the induction hypothesis to the above inequality, we obtain

$$V_k(\mathbf{m}|L_k^*(\mathbf{m}), L_k^{\pi_0}(\mathbf{m})) \leq \mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}} + \mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m}) > 0\}} + 2V_{k-1,\pi_0}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})}). \quad (12)$$

Finally, taking expectations on both sides of the above inequality leads to

$$V_k(\mathbf{m}) \leq E[\mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}}] + E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m}) > 0\}}] + 2E[V_{k-1,\pi_0}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})})]. \quad (13)$$

Now note that $E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m}) > 0\}}] \geq E[\mathbb{1}_{\{L_k^*(\mathbf{m}) > 0\}}]$ by the definition of the greedy policy, and hence we arrive at

$$V_k(\mathbf{m}) \leq 2E[\mathbb{1}_{\{L_k^{\pi_0}(\mathbf{m}) > 0\}}] + 2E[V_{k-1,\pi_0}(\mathbf{m} - \mathbf{e}_{L_k^{\pi_0}(\mathbf{m})})] = 2V_{k,\pi_0}(\mathbf{m}). \quad (14)$$

\square

\square

C Proof of Theorem 2

We first present the scaling conditions under which Theorem 2 holds. Consider a base system in which $I = I_0$, $J = J_0$, $\lambda_i = \lambda_{i,0}$, $N = N_0$ and \mathbf{b} which is a 1 by J_0 vector denoting the initial capacity. For ease of presentation, we assume $b_j = b$, $\forall j$ (unequal capacity cases can be analyzed in a similar fashion). Let Λ_i be the total number of slot types accepted by patient type i , i.e., $\Lambda_i = \sum_{j=1}^J \Omega_{ij}$. Define a set \mathcal{D} to be the set of patient types who only accept a single slot type, i.e., $\mathcal{D} = \{i : \Lambda_i = 1\}$; note that $|\mathcal{D}| \leq J$. Then its complement \mathcal{M} contains patient types who accept more than one slot types; we have $\mathcal{D} \cup \mathcal{M} = \{1, 2, \dots, I\}$.

We consider a sequence of systems indexed by J , the number of slot types. The size and dimension of the system increase in J . Specifically, the scaling of these systems obeys the following conditions. For convenience, we say a function $f(J) = O(J^k)$ (or $f(J)$ is in the order of J^k where k is an integer), if and

only if $\lim_{J \rightarrow \infty} f(J)/J^k = f_0 > 0$.

$$I^J = O(J), \quad (15)$$

$$N^J = O(J), \quad (16)$$

$$\mathbf{b}^J = \underbrace{(b, b, \dots, b)}_J, \quad (17)$$

$$\sum_i \Omega_{ij}^J \geq 1, \quad \forall j = 1, 2, \dots, J, \quad (18)$$

$$\sum_j \Omega_{ij}^J \geq 1, \quad \forall i = 1, 2, \dots, I^J, \quad (19)$$

$$\Lambda_i^J \leq B, \quad \forall i = 1, 2, \dots, I^J, \quad (20)$$

$$\min_i \lambda_i^J = O(J^{-1}), \quad (21)$$

$$\lim_{J \rightarrow \infty} \sum_{i \in \mathcal{M}^J} \lambda_i^J = 0, \quad (22)$$

$$\sum_i \lambda_i^J = 1. \quad (23)$$

Conditions (15) and (16) say that the number of patient types I^J and the number of periods to go N^J increase in the order of J . Conditions (17) implies that the total capacity also increases in the order of J . Conditions (18) and (19) ensure that $\forall i$, there is some j such that $\Omega_{ij}^J = 1$ and that $\forall j$, there is some i such that $\Omega_{ij}^J = 1$. Condition (20) means that each patient type can accept no more than $B > 0$ slot types. Conditions (21) ensures that the decrease of the arrival probability is in the order of $1/J$; it also implies that there exists $c_0 > 0$ such that

$$\lim_{J \rightarrow \infty} N^J [\min_i \lambda_i^J] = c_0 > 0. \quad (24)$$

Condition (22) suggests that the size of the set, \mathcal{M}^J , cannot increase too fast; as λ_i^J decreases in an order of $1/J$, \mathcal{M}^J cannot increase in an order of J . (23) avoids triviality and requires that the total arrival probability is 1 for all systems. To give a nontrivial example that obeys the last few conditions (21)-(23) imposed on the arrival probabilities, we could have $\lambda_i^J = 1/I^J = 1/f(J)$ and $|\mathcal{M}^J| \sim o(J)$.

We note that in the regime above, the ratio between total expected demand and total capacity is N^J/mJ , which converges to a constant as $J \rightarrow \infty$. We show below that in this regime, the greedy policy π_0 that offers all available slot types is asymptotically optimal as $J \rightarrow \infty$ in terms of the percentage optimality gap.

We are now ready to prove Theorem 2.

Proof. Proof. We first construct an upper bound on the value function $V_n(\mathbf{m})$ for $\mathbf{m} \geq 0$, and it suffices to show that the percentage gap of the value function under the greedy policy and this upper bound approaches 0 as $J \rightarrow \infty$. To construct the upper bound, we use an existing result from Zhang and Cooper (2005). We start by defining $\bar{q}_j = \sum_{i=1}^I \lambda_i \Omega_{ij}$. Note that \bar{q}_j is an upper bound on the demand rate to slot type j regardless of the system state and action chosen. Proposition 2 in Zhang and Cooper (2005) suggests the following upper bound $\bar{V}_n(\mathbf{m})$:

$$\bar{V}_n(\mathbf{m}) = \sum_{j=1}^J \bar{V}_n^j(m_j) \geq V_n(\mathbf{m}), \quad (25)$$

where $\bar{V}_n^j(m_j)$ is the maximum number of patients booked in a system with a single slot type j , initial capacity m , n periods to go and patient arrival probability per period being \bar{q}_j . Note that the LHS above is the sum of value functions of J independent single dimensional MDPs, and for each MDP the greedy policy is optimal. Specifically, we have

$$\bar{V}_n^j(m_j) = \bar{q}_j(1 + \bar{V}_{n-1}^j(m_j - 1)) + (1 - \bar{q}_j)\bar{V}_{n-1}^j(m_j).$$

It is not difficult to see that $\bar{V}_n^j(m_j) = E[\min(m_j, \bar{D}_n^j)]$, where $\bar{D}_n^j \sim \text{Bin}(n, \bar{q}_j)$.

Let $\underline{\lambda}_i^J = \min_i \lambda_i^J$. It follows that $\forall J$, $E[\min(m_j, \text{Bin}(N^J, \bar{q}_j^J))] \geq E[\min(m_j, \text{Bin}(N^J, \underline{\lambda}_i^J))]$ (because each patient type accepts at least one slot type; see condition (19)). As $J \rightarrow \infty$, $\text{Bin}(N^J, \underline{\lambda}_i^J)$ converges to a Poisson random variable with mean $\lim_{J \rightarrow \infty} N^J \underline{\lambda}_i^J = c_0$ (conditions (16), (21) and (24)). Thus, we have

$$\lim_{J \rightarrow \infty} \frac{\bar{V}_{N^J}(\mathbf{b})}{J} = \lim_{J \rightarrow \infty} \frac{\sum_{j=1}^J E[\min(b, \bar{D}_{n,j}^J)]}{J} \geq E[\min(b, \text{Poi}(c_0))].$$

On the other hand, $\bar{V}_{N^J}(\mathbf{b})$ is bounded above by bJ , the total capacity in the system. That is,

$$\lim_{J \rightarrow \infty} \frac{\bar{V}_{N^J}(\mathbf{b})}{J} \leq b.$$

It follows that $\bar{V}_{N^J}(\mathbf{b}) = O(J)$.

To facilitate our proof, we create an intermediate bound $\bar{V}_{n,\pi_0}(\mathbf{m})$ such that $V_{n,\pi_0}(\mathbf{m}) \leq \bar{V}_{n,\pi_0}(\mathbf{m}) \leq \bar{V}_n(\mathbf{m})$, $\forall n, \mathbf{m} \geq 0$ as follows. Define $\bar{J}(\mathbf{m}) = \{k = 1, 2, \dots, J : m_k > 0\}$ be the set of slot types with positive capacity. Let $\Lambda_i(\mathbf{m}) = \sum_{j \in \bar{J}(\mathbf{m})} \Omega_{ij}$. Then

$$q_j(\mathbf{m}) = \begin{cases} \sum_{i=1}^I \lambda_i \Omega_{ij} / \Lambda_i(\mathbf{m}), & \text{if } \Lambda_i(\mathbf{m}) > 0, \\ 0, & \text{otherwise} \end{cases}$$

is the demand rate faced by slot type j under the greedy policy π_0 . Recall that $\bar{q}_j \geq q_j(\mathbf{m})$, $\forall \mathbf{m} \geq 0$, and we have

$$\begin{aligned} V_{n,\pi_0}(\mathbf{m}) &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})(1 + V_{n-1,\pi_0}(\mathbf{m} - \mathbf{e}_j)) + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}))V_{n-1,\pi_0}(\mathbf{m}) \\ &\leq \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})(1 + \bar{V}_{n-1}(\mathbf{m} - \mathbf{e}_j)) + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}))\bar{V}_{n-1}(\mathbf{m}) \end{aligned} \quad (26)$$

$$\begin{aligned} &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) + \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})(\bar{V}_{n-1}(\mathbf{m} - \mathbf{e}_j) - \bar{V}_{n-1}(\mathbf{m})) + \bar{V}_{n-1}(\mathbf{m}) \\ &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) + \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})(\bar{V}_{n-1}^j(m_j - 1) - \bar{V}_{n-1}^j(m_j)) + \sum_{j \in \bar{J}(\mathbf{m})} \bar{V}_{n-1}^j(m_j) \end{aligned} \quad (27)$$

$$\begin{aligned} &= \sum_{j \in \bar{J}(\mathbf{m})} [q_j(\mathbf{m})(1 + \bar{V}_{n-1}^j(m_j - 1)) + (1 - q_j(\mathbf{m}))\bar{V}_{n-1}^j(m_j)] \\ &= \bar{V}_{n,\pi_0}(\mathbf{m}) \end{aligned} \quad (28)$$

$$\leq \sum_{j \in \bar{J}(\mathbf{m})} [\bar{q}_j(1 + \bar{V}_{n-1}^j(m_j - 1)) + (1 - \bar{q}_j)\bar{V}_{n-1}^j(m_j)] \quad (29)$$

$$= \bar{V}_n(\mathbf{m}). \quad (30)$$

In the inequalities above, (26) follows (25), (28) defines $\bar{V}_{n,\pi_0}(\mathbf{m})$, (29) is resulted from the fact that $\bar{q}_j \geq q_j(\mathbf{m})$ and $1 + \bar{V}_{n-1}^j(m_j - 1) - \bar{V}_{n-1}^j(m_j) \geq 0$, and (27) and (30) follow from that $\bar{V}_{n-1}^j(m_j) = 0$ if $m_j = 0$.

For ease of presentation, we suppress the system index J in the following proof when the context is clear. Let $\Delta_q^J = \sum_{j=1}^J \bar{q}_j - \sum_{j=1}^J q_j(\mathbf{e})$ where \mathbf{e} is a J dimensional unit vector, and $\Delta_q^J(\mathbf{m}) = \sum_{j \in \bar{J}(\mathbf{m})} \bar{q}_j - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})$. Note that $\Delta_q^J(\mathbf{m}) \leq \Delta_q^J$, $\forall \mathbf{m} \geq 0$. Next, we show that for a fixed J , $\bar{V}_n(\mathbf{m}) - V_{n,\pi_0}(\mathbf{m}) \leq n\Delta_q^J$, $\forall n = 1, 2, \dots, N^J$, $\forall \mathbf{m} \geq 0$ by induction. When $n = 1$, we have $\bar{V}_1(\mathbf{m}) - V_{1,\pi_0}(\mathbf{m}) = \Delta_q^J(\mathbf{m}) \leq \Delta_q^J$, $\forall \mathbf{m} \geq 0$. Suppose the desired inequality holds up to $n - 1$ and consider the case n . We have $\forall \mathbf{m} \geq 0$,

$$\begin{aligned} &\bar{V}_n(\mathbf{m}) - V_{n,\pi_0}(\mathbf{m}) \\ &= (\bar{V}_n(\mathbf{m}) - \bar{V}_{n,\pi_0}(\mathbf{m})) + (\bar{V}_{n,\pi_0}(\mathbf{m}) - V_{n,\pi_0}(\mathbf{m})) \\ &= \sum_{j \in \bar{J}(\mathbf{m})} (\bar{q}_j - q_j(\mathbf{m}))(1 + \bar{V}_{n-1}^j(m_j - 1) - \bar{V}_{n-1}^j(m_j)) \\ &\quad + \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})(\bar{V}_{n-1}(\mathbf{m} - \mathbf{e}_j) - V_{n-1,\pi_0}(\mathbf{m} - \mathbf{e}_j)) + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}))[\bar{V}_{n-1}(\mathbf{m}) - V_{n-1,\pi_0}(\mathbf{m})] \\ &\leq \Delta_q^J + \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})[(n-1)\Delta_q^J] + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}))[(n-1)\Delta_q^J] \\ &= n\Delta_q^J, \end{aligned}$$

where the last inequality follows from that fact that $1 + \bar{V}_{n-1}^j(m_j - 1) - \bar{V}_{n-1}^j(m_j) \leq 1$ and the induction hypothesis. This completes the induction proof.

We claim that

$$\lim_{J \rightarrow \infty} \Delta_q^J = 0, \quad (31)$$

and

$$V_{N,\pi_0}(\mathbf{b}) = O(J). \quad (32)$$

Then, it follows that

$$0 \leq \lim_{J \rightarrow \infty} \frac{V_N(\mathbf{b}) - V_{N,\pi_0}(\mathbf{b})}{V_{N,\pi_0}(\mathbf{b})} \leq \lim_{J \rightarrow \infty} \frac{\bar{V}_N(\mathbf{b}) - V_{N,\pi_0}(\mathbf{b})}{V_{N,\pi_0}(\mathbf{b})} \leq \lim_{J \rightarrow \infty} \frac{N^J \Delta_q^J}{V_{N,\pi_0}(\mathbf{b})} = \lim_{J \rightarrow \infty} \frac{N^J \Delta_q^J / J}{V_{N,\pi_0}(\mathbf{b}) / J} = 0,$$

where the second equality follows from (25), and the last equality results from Condition (16), our claims (31) and (32). The inequality chain above yields the desired result.

Finally, it is left to prove claims (31) and (32). To show (31), we first define $q_{ij} = \lambda_i \Omega_{ij} / \Lambda_i$ and $\bar{q}_{ij} = \lambda_i \Omega_{ij}$. Then,

$$\begin{aligned} \Delta_q^J &= \sum_{j=1}^J \bar{q}_j - \sum_{j=1}^J q_j = \sum_{i=1}^I \sum_{j=1}^J (\bar{q}_{ij} - q_{ij}) = \sum_{i \in \mathcal{D}} \sum_{j=1}^J (\bar{q}_{ij} - q_{ij}) + \sum_{i \in \mathcal{M}} \sum_{j=1}^J (\bar{q}_{ij} - q_{ij}) \\ &= \sum_{i \in \mathcal{M}} \sum_{j=1}^J (\bar{q}_{ij} - q_{ij}) \leq \sum_{i \in \mathcal{M}} \sum_{j=1}^J \lambda_i \Omega_{ij} \frac{B-1}{B} = \sum_{i \in \mathcal{M}} \lambda_i \frac{B-1}{B} \sum_{j=1}^J \Omega_{ij} \\ &= \sum_{i \in \mathcal{M}} \lambda_i \frac{B-1}{B} \Lambda_i \leq \sum_{i \in \mathcal{M}} \lambda_i (B-1), \end{aligned}$$

where both inequalities follow from Condition (20). Then, Condition (22) suggests that claim (31) holds.

To prove claim (32), we first define $\underline{q}_j = \sum_{i=1}^I \lambda_i \frac{\Omega_{ij}}{J}$. Let

$$\begin{aligned} \underline{V}_n(\mathbf{m}) &= \sum_{j \in \bar{J}(\mathbf{m})} \underline{V}_n^j(m_j) = \sum_{j \in \bar{J}(\mathbf{m})} [\underline{q}_j (1 + \underline{V}_{n-1}^j(m_j - 1)) + (1 - \underline{q}_j) \underline{V}_{n-1}^j(m_j)] \\ &= \sum_{j \in \bar{J}(\mathbf{m})} [\underline{q}_j (1 + \underline{V}_{n-1}^j(m_j - 1) - \underline{V}_{n-1}^j(m_j)) + \underline{V}_{n-1}(\mathbf{m})]. \end{aligned}$$

We next show that $\underline{V}_n(\mathbf{m}) \leq V_{n,\pi_0}(\mathbf{m})$, $\forall \mathbf{m} \geq 0$ by induction. When $n = 1$, it is clear that $\underline{V}_1(\mathbf{m}) \leq V_{1,\pi_0}(\mathbf{m})$ because $\underline{q}_j \leq q_j(\mathbf{m})$ as long as $m_j > 0$. Assume the hypothesis holds up to $n - 1$ and now consider the case n . Then,

$$\begin{aligned} V_{n,\pi_0}(\mathbf{m}) &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) (1 + V_{n-1,\pi_0}(\mathbf{m} - \mathbf{e}_j)) + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})) V_{n-1,\pi_0}(\mathbf{m}) \\ &\geq \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) (1 + \underline{V}_{n-1}(\mathbf{m} - \mathbf{e}_j)) + (1 - \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m})) \underline{V}_{n-1}(\mathbf{m}) \\ &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) (1 + \underline{V}_{n-1}(\mathbf{m} - \mathbf{e}_j) - \underline{V}_{n-1}(\mathbf{m})) + \underline{V}_{n-1}(\mathbf{m}) \\ &= \sum_{j \in \bar{J}(\mathbf{m})} q_j(\mathbf{m}) (1 + \underline{V}_{n-1}^j(m_j - 1) - \underline{V}_{n-1}^j(m_j)) + \underline{V}_{n-1}(\mathbf{m}) \\ &\geq \sum_{j \in \bar{J}(\mathbf{m})} \underline{q}_j (1 + \underline{V}_{n-1}^j(m_j - 1) - \underline{V}_{n-1}^j(m_j)) + \underline{V}_{n-1}(\mathbf{m}) \\ &= \underline{V}_n(\mathbf{m}), \end{aligned}$$

where the first inequality follows the induction hypothesis and the last inequality is resulted from the fact that $q_j(\mathbf{m}) \geq \underline{q}_j$, $\forall j \in \bar{J}(\mathbf{m})$ and that $1 + \underline{V}_{n-1}^j(m_j - 1) - \underline{V}_{n-1}^j(m_j) \geq 0$.

Now, we note that $\underline{V}_n^j(m_j) = E[\min(m_j, \underline{D}_n^j)]$, where $\underline{D}_n^j \sim \text{Bin}(n, \underline{q}_j)$. It follows a similar argument above that $\underline{V}_{N^J}(\mathbf{b})$ is in the order of J . Thus, $V_{N^J,\pi_0}(\mathbf{b})$, which is bounded by the total capacity in the system bJ , must be in the order of J , proving (32) and completing the whole proof. \square

\square

D Proof of Proposition 1

Proof. Proof. We focus on the W model instance here, as the N Model instance is a special case of this. For $n \geq 1$ and any system state $(x, y) \geq (1, 1)$, the optimality equation for the W model instance reads.

$$V_n(x, y) = \max \left\{ \begin{array}{l} 1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{n-1}(x-1, y) + (\frac{1}{2}\lambda_2 + \lambda_3)V_{n-1}(x, y-1) + \lambda_0 V_{n-1}(x, y), \\ (1 - \lambda_3 - \lambda_0) + (\lambda_3 + \lambda_0)V_{n-1}(x, y) + (\lambda_1 + \lambda_2)V_{n-1}(x-1, y), \\ (1 - \lambda_1 - \lambda_0) + (\lambda_1 + \lambda_0)V_{n-1}(x, y) + (\lambda_2 + \lambda_3)V_{n-1}(x, y-1) \end{array} \right\}, \quad (33)$$

where the three terms in the max operator correspond to the action of offering slot types $\{1, 2\}$, $\{1\}$ and $\{2\}$, respectively. For the boundary conditions, it is easy to see that $V_0(x, y) = 0$ regardless of x and y . When one type of the slots are depleted, it is optimal to offer the other type of the slots. To calculate $V_n(x, 0)$, note that type 1 slots are accepted only by type 1 and type 2 patients and the number of type 1 and type 2 patients in the last n patients yet to come has a binomial distribution with parameters n and $\lambda_1 + \lambda_2$. Denote this random variable by $X_1 \sim \text{Bin}(n, \lambda_1 + \lambda_2)$. It follows that

$$V_n(x, 0) = \mathbf{E}(\min\{x, X_1\}) = \sum_{k=0}^n \min(x, k) \binom{n}{k} (\lambda_1 + \lambda_2)^k (1 - \lambda_1 - \lambda_2)^{n-k}. \quad (34)$$

Similarly, with $X_2 \sim \text{Bin}(n, \lambda_2 + \lambda_3)$

$$V_n(0, y) = \mathbf{E}(\min\{y, X_2\}) = \sum_{k=0}^n \min(y, k) \binom{n}{k} (\lambda_2 + \lambda_3)^k (1 - \lambda_2 - \lambda_3)^{n-k}. \quad (35)$$

For ease of presentation, we define $\Delta_n^{ij}(x, y)$ to be the difference of the i th and j th terms in the max operator (33) above, $i, j \in \{1, 2, 3\}$. In particular, we have

$$\Delta_n^{12}(x, y) = \lambda_3 - \frac{1}{2}\lambda_2 V_{n-1}(x-1, y) + (\frac{1}{2}\lambda_2 + \lambda_3)V_{n-1}(x, y-1) - \lambda_3 V_{n-1}(x, y), \quad (36)$$

and

$$\Delta_n^{13}(x, y) = \lambda_1 - \frac{1}{2}\lambda_2 V_{n-1}(x, y-1) + (\frac{1}{2}\lambda_2 + \lambda_1)V_{n-1}(x-1, y) - \lambda_1 V_{n-1}(x, y). \quad (37)$$

It suffices to show that $\Delta_n^{12}(x, y), \Delta_n^{13}(x, y) \geq 0$ for any $x, y \geq 1$ (the case when x or y equals 0 is trivial as it meets the boundary conditions discussed above; see (34) and (35)). We use induction below to prove this. When $n = 1$, it is a trivial proof as it is optimal to offer all available slots with one period left. Suppose that (36) and (37) hold up to $n = k$ and for any $x, y \geq 1$. Now, consider $n = k + 1$ and $x, y \geq 1$. We have four cases to check: (1) $x = y = 1$; (2) $y = 1$ and $x \geq 2$; (3) $x = 1$ and $y \geq 2$; and (4) $x, y \geq 2$. We start with

case (1) and evaluate the term $\Delta_{k+1}^{13}(x, 1)$ below.

$$\begin{aligned}
\Delta_{k+1}^{13}(1, 1) &= \lambda_1 + (\lambda_1 + \frac{1}{2}\lambda_2)V_k(0, 1) - \frac{1}{2}\lambda_2V_k(1, 0) - \lambda_1V_k(1, 1) \\
&= \lambda_1 + (\lambda_1 + \frac{1}{2}\lambda_2)[1 - (\lambda_1 + \lambda_0)^k] - \frac{1}{2}\lambda_2[1 - (\lambda_3 + \lambda_0)^k] \\
&\quad - \lambda_1[1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(0, 1) + (\lambda_3 + \frac{1}{2}\lambda_2)V_{k-1}(1, 0) + \lambda_0V_{k-1}(1, 1)] \\
&= \lambda_0\Delta_k^{13}(1, 1) + (\lambda_1 + \frac{1}{2}\lambda_2)[1 - (\lambda_1 + \lambda_0)^k] - \frac{1}{2}\lambda_2[1 - (\lambda_3 + \lambda_0)^k] \\
&\quad - \lambda_1[(\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(0, 1) + (\lambda_3 + \frac{1}{2}\lambda_2)V_{k-1}(1, 0)] \\
&\quad - \lambda_0(\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(0, 1) + \frac{1}{2}\lambda_0\lambda_2V_{k-1}(1, 0) \\
&= \lambda_0\Delta_k^{13}(1, 1) + (\lambda_1 + \frac{1}{2}\lambda_2)[1 - (\lambda_1 + \lambda_0)] - \frac{1}{2}\lambda_2[1 - (\lambda_3 + \lambda_0)^k] \\
&\quad + [\frac{1}{2}\lambda_0\lambda_2 - \lambda_1(\lambda_3 + \frac{1}{2}\lambda_2)][1 - (\lambda_3 + \lambda_0)^{k-1}] \\
&= \lambda_0\Delta_k^{13}(1, 1) + [\frac{1}{2}\lambda_2(\lambda_3 + \lambda_0) - \frac{1}{2}\lambda_0\lambda_2 + \lambda_1(\lambda_3 + \frac{1}{2}\lambda_2)](\lambda_3 + \lambda_0)^{k-1} \\
&= \lambda_0\Delta_k^{13}(1, 1) + [\frac{1}{2}\lambda_2(\lambda_1 + \lambda_3) + \lambda_1\lambda_3](\lambda_3 + \lambda_0)^{k-1} \geq 0
\end{aligned}$$

where the second equality follow from (34), (35) and the induction hypothesis. Observing the symmetry, we can show $\Delta_{k+1}^{12}(1, 1) \geq 0$.

We now study case (2). We can evaluate the term $\Delta_{k+1}^{13}(x, 1)$ as below.

$$\begin{aligned}
\Delta_{k+1}^{13}(x, 1) &= \lambda_1 + \lambda_1V_k(x-1, 1) - \lambda_1V_k(x, 1) + \frac{1}{2}\lambda_2V_k(x-1, 1) - \frac{1}{2}\lambda_2V_k(x, 0) \\
&= \lambda_1 + \lambda_1[1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(x-2, 1) + (\frac{1}{2}\lambda_2 + \lambda_3)V_{k-1}(x-1, 0) + \lambda_0V_{k-1}(x-1, 1)] \\
&\quad - \lambda_1[1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(x-1, 1) + (\frac{1}{2}\lambda_2 + \lambda_3)V_{k-1}(x, 0) + \lambda_0V_{k-1}(x, 1)] \\
&\quad + \frac{1}{2}\lambda_2[1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{k-1}(x-2, 1) + (\frac{1}{2}\lambda_2 + \lambda_3)V_{k-1}(x-1, 0) + \lambda_0V_{k-1}(x-1, 1)] \\
&\quad - \frac{1}{2}\lambda_2[1 - \lambda_3 - \lambda_0 + (\lambda_1 + \lambda_2)V_{k-1}(x-1, 0) + \lambda_3V_{k-1}(x, 0) + \lambda_0V_{k-1}(x, 0)],
\end{aligned}$$

where the second equality follows from the induction hypothesis. Note that $\lambda_1 = \lambda_1(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)$. We can continue the equality chain above as follows.

$$\begin{aligned}
&\Delta_{k+1}^{13}(x, 1) \\
&= (\lambda_1 + \frac{1}{2}\lambda_2)[\lambda_1 + \lambda_1V_{k-1}(x-2, 1) - \lambda_1V_{k-1}(x-1, 1) + \frac{1}{2}\lambda_2V_{k-1}(x-2, 1) - \frac{1}{2}\lambda_2V_{k-1}(x-1, 0)] \\
&\quad + \lambda_0[\lambda_1 + \lambda_1V_{k-1}(x-1, 1) - \lambda_1V_{k-1}(x, 1) + \frac{1}{2}\lambda_2V_{k-1}(x-1, 1) - \frac{1}{2}\lambda_2V_{k-1}(x, 0)] \\
&\quad + (\frac{1}{2}\lambda_2 + \lambda_3)[\lambda_1 + \lambda_1V_{k-1}(x-1, 0) - \lambda_1V_{k-1}(x, 0) + \frac{1}{2}\lambda_2V_{k-1}(x-1, 0)] \\
&\quad + \frac{1}{2}\lambda_2\lambda_3 - \frac{1}{2}\lambda_2\frac{1}{2}\lambda_2V_{k-1}(x-1, 0) - \frac{1}{2}\lambda_2\lambda_3V_{k-1}(x, 0) \\
&= (\lambda_1 + \frac{1}{2}\lambda_2)\Delta_k^{13}(x-1, 1) + \lambda_0\Delta_k^{13}(x, 1) + (\frac{1}{2}\lambda_2\lambda_1 + \lambda_3\lambda_1 + \frac{1}{2}\lambda_2\lambda_3)[1 + V_{k-1}(x-1, 0) - V_{k-1}(x, 0)] \geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis (37) and Lemma 3. Following a similar proof, we can show that $\Delta_{k+1}^{12}(x, 1), \Delta_{k+1}^{12}(1, y), \Delta_{k+1}^{13}(1, y) \geq 0$ for $x, y \geq 2$.

Finally, we consider case (4) and evaluate the term $\Delta_{k+1}^{13}(x, y)$ below.

$$\begin{aligned}
& \Delta_{k+1}^{13}(x, y) \\
&= \lambda_1 + \lambda_1 V_k(x-1, y) - \lambda_1 V_k(x, y) + \frac{1}{2} \lambda_2 V_k(x-1, y) - \frac{1}{2} \lambda_2 V_k(x, y-1) \\
&= \lambda_1 + \lambda_1 [1 - \lambda_0 + (\lambda_1 + \frac{1}{2} \lambda_2) V_{k-1}(x-2, y) + (\frac{1}{2} \lambda_2 + \lambda_3) V_{k-1}(x-1, y-1) + \lambda_0 V_{k-1}(x-1, y)] \\
&\quad - \lambda_1 [1 - \lambda_0 + (\lambda_1 + \frac{1}{2} \lambda_2) V_{k-1}(x-1, y) + (\frac{1}{2} \lambda_2 + \lambda_3) V_{k-1}(x, y-1) + \lambda_0 V_{k-1}(x, y)] \\
&\quad + \frac{1}{2} \lambda_2 [1 - \lambda_0 + (\lambda_1 + \frac{1}{2} \lambda_2) V_{k-1}(x-2, y) + (\frac{1}{2} \lambda_2 + \lambda_3) V_{k-1}(x-1, y-1) + \lambda_0 V_{k-1}(x-1, y)] \\
&\quad - \frac{1}{2} \lambda_2 [1 - \lambda_0 + (\lambda_1 + \frac{1}{2} \lambda_2) V_{k-1}(x-1, y-1) + (\frac{1}{2} \lambda_2 + \lambda_3) V_{k-1}(x, y-2) + \lambda_0 V_{k-1}(x, y-1)],
\end{aligned}$$

where the second equality follows from the induction hypothesis. Recall that $\sum_{i=0}^3 \lambda_i = 1$ and thus $\lambda_1 = \lambda_1(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)$. We can continue the equality chain above as follows.

$$\begin{aligned}
\Delta_{k+1}^{13}(x, y) &= (\lambda_1 + \frac{1}{2} \lambda_2) [\lambda_1 + \lambda_1 V_{k-1}(x-2, y) - \lambda_1 V_{k-1}(x-1, y) \\
&\quad + \frac{1}{2} \lambda_2 V_{k-1}(x-2, y) - \frac{1}{2} \lambda_2 V_{k-1}(x-1, y-1)] \\
&\quad + (\frac{1}{2} \lambda_2 + \lambda_3) [\lambda_1 + \lambda_1 V_{k-1}(x-1, y-1) - \lambda_1 V_{k-1}(x, y-1) \\
&\quad + \frac{1}{2} \lambda_2 V_{k-1}(x-1, y-1) - \frac{1}{2} \lambda_2 V_{k-1}(x, y-2)] \\
&\quad + \lambda_0 [\lambda_1 - \frac{1}{2} \lambda_2 V_{k-1}(x, y-1) + (\lambda_1 + \frac{1}{2} \lambda_2) V_{k-1}(x-1, y) - \lambda_1 V_{k-1}(x, y)] \\
&= (\lambda_1 + \frac{1}{2} \lambda_2) \Delta_k^{13}(x-1, y) + (\frac{1}{2} \lambda_2 + \lambda_3) \Delta_k^{12}(x, y-1) + \lambda_0 \Delta_k^{13}(x, y) \geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis. Using similar arguments, we can show that $\Delta_{k+1}^{12}(x, y) \geq 0$ for $x, y \geq 2$. Combining the four cases above, we prove the desired result. \square \square

E Proof of Proposition 2

Before we prove Proposition 2, we first present an auxiliary result.

Lemma 4. *Consider the “M” network and let $n \in \mathbb{N}$. Then*

$$V_n(0, m_2, m_3 - 1) \geq V_n(0, m_2 - 1, m_3), \quad m_2 \geq 1, \quad m_3 \geq 1, \quad (38)$$

$$V_n(m_1 - 1, m_2, 0) \geq V_n(m_1, m_2 - 1, 0), \quad m_1 \geq 1, \quad m_2 \geq 1. \quad (39)$$

Proof. Proof. We will prove (38) by induction; this immediately implies (39) due to symmetry.

First, we can see by inspection that

$$V_1(0, m_2, m_3 - 1) = 1 - \lambda_0 \geq V_1(0, m_2 - 1, m_3). \quad (40)$$

Now, let $t \in \mathbb{N}$ and assume that (38) holds for all $n \leq t$. In order to show that (38) holds for $n = t + 1$ as well, first observe that for $m_1 = 0$, the M model reduces to the N model, and by Proposition 1 we know that it is optimal to offer all slots:

$$\begin{aligned}
V_n(0, m_2, m_3) &= (1 - \lambda_0) + (\lambda_1 + \frac{1}{2} \lambda_2) V_{n-1}(0, m_2 - 1, m_3) + \frac{1}{2} \lambda_2 V_{n-1}(0, m_2, m_3 - 1) \\
&\quad + \lambda_0 V_{n-1}(0, m_2, m_3), \quad m_2 \geq 1, \quad m_3 \geq 1, \quad n \in \mathbb{N},
\end{aligned} \quad (41)$$

and

$$V_n(0, m_2, 0) = (1 - \lambda_0) + (1 - \lambda_0) V_{n-1}(0, m_2 - 1, 0) + \lambda_0 V_{n-1}(0, m_2, 0), \quad m_2 \geq 1, \quad n \in \mathbb{N}. \quad (42)$$

We first prove that (38) holds for $m_2 \geq 2$ and $m_3 \geq 2$, and treat the boundary cases separately. Using (41) we can write

$$\begin{aligned}
& V_{t+1}(0, m_2, m_3 - 1) \\
&= (1 - \lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0, m_2 - 1, m_3 - 1) + \frac{1}{2}\lambda_2 V_t(0, m_2, m_3 - 2) + \lambda_0 V_t(0, m_2, m_3 - 1) \\
&\geq (1 - \lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0, m_2 - 2, m_3) + \frac{1}{2}\lambda_2 V_t(0, m_2 - 1, m_3 - 1) + \lambda_0 V_t(0, m_2 - 1, m_3) \\
&= V_{t+1}(0, m_2 - 1, m_3).
\end{aligned} \tag{43}$$

Here we use the induction hypothesis (38) (with $n = t$) for the inequality, and use (41) for the second equality.

For the case $m_2 \geq 2$ and $m_3 = 1$ we use (42) to obtain

$$\begin{aligned}
& V_{t+1}(0, m_2, 0) \\
&= (1 - \lambda_0) + (1 - \lambda_0)V_t(0, m_2 - 1, 0) + \lambda_0 V_t(0, m_2, 0)
\end{aligned} \tag{44}$$

$$\geq (1 - \lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0, m_2 - 2, 1) + \frac{1}{2}\lambda_2 V_t(0, m_2 - 1, 0) + \lambda_0 V_t(0, m_2 - 1, 1) \tag{45}$$

$$= V_{t+1}(0, m_2 - 1, 1), \tag{46}$$

where the inequality follows from the induction hypothesis (38), and the final equality from our knowledge on the optimal control for $n = t + 1$, see (41).

For the case $m_2 = 1$ and $m_3 \geq 2$ we write, using (41),

$$\begin{aligned}
& V_{t+1}(0, 1, m_3 - 1) \\
&= (1 - \lambda_0) + (\lambda_1 + \frac{1}{2}\lambda_2)V_t(0, 0, m_3 - 1) + \frac{1}{2}\lambda_2 V_t(0, 1, m_3 - 2) + \lambda_0 V_t(0, 1, m_3 - 1) \\
&= (1 - \lambda_0 - \lambda_1) + \frac{1}{2}\lambda_2 V_t(0, 0, m_3 - 1) + \frac{1}{2}\lambda_2 V_t(0, 1, m_3 - 2) + \lambda_1(1 + V_t(0, 0, m_3 - 1)) \\
&\quad + \lambda_0 V_t(0, 1, m_3 - 1) \\
&\geq (1 - \lambda_0 - \lambda_1) + \lambda_2 V_t(0, 0, m_3 - 1) + (\lambda_0 + \lambda_1)V_t(0, 0, m_3) \\
&= V_{t+1}(0, 0, m_3).
\end{aligned} \tag{47}$$

For the second inequality, we use the induction hypothesis (38) and apply Lemma 3(ii) to show that $1 + V_t(0, 0, m_3 - 1) \geq V_t(0, 0, m_3)$.

The case $m_2 = m_3 = 1$ we can do directly, by observing that

$$V_{t+1}(0, 1, 0) = 1 - (1 - \lambda_0)^{t+1} \geq 1 - (1 - \lambda_0 - \lambda_2)^{t+1} = V_{t+1}(0, 0, 1), \tag{48}$$

completing the proof. \square

With Lemma 4, we can prove Proposition 2 now.

Proof. Proof of Proposition 2. From the boundary conditions, it is easy to see that $V_0(\mathbf{m}) = 0$ regardless of \mathbf{m} . When $m_2 = 0$ the problem degenerates into two separate problems with a single patient type and single slot type where the straightforward optimal decision is to offer all slots to patients. When either $m_1 = 0$ or $m_3 = 0$, the problem reduces to an “N” model and it is optimal to offer all available slots (see Proposition 1). Thus, what remains to be shown is that when none of the slots are depleted, it is optimal to offer type-1 and type-3 slots, but block type-2 slots.

Throughout this proof we assume that $\mathbf{m} \geq (1, 1, 1)$, unless stated otherwise. In this case, the Bellman

equation can be written as

$$V_n(\mathbf{m}) = \max \left\{ \begin{array}{l} 1 - \lambda_0 + \frac{1}{2}\lambda_1 V_{n-1}(\mathbf{m} - \mathbf{e}_1) + \frac{1}{2}(\lambda_1 + \lambda_2)V_{n-1}(\mathbf{m} - \mathbf{e}_2) + \frac{1}{2}\lambda_2 V_{n-1}(\mathbf{m} - \mathbf{e}_3) \\ + \lambda_0 V_{n-1}(\mathbf{m}), \\ 1 - \lambda_0 + \frac{1}{2}\lambda_1 V_{n-1}(\mathbf{m} - \mathbf{e}_1) + (\frac{1}{2}\lambda_1 + \lambda_2)V_{n-1}(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_{n-1}(\mathbf{m}), \\ 1 - \lambda_0 + \lambda_1 V_{n-1}(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_{n-1}(\mathbf{m} - \mathbf{e}_3) + \lambda_0 V_{n-1}(\mathbf{m}), \\ 1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_{n-1}(\mathbf{m} - \mathbf{e}_2) + \frac{1}{2}\lambda_2 V_{n-1}(\mathbf{m} - \mathbf{e}_3) + \lambda_0 V_{n-1}(\mathbf{m}), \\ \lambda_1 + \lambda_1 V_{n-1}(\mathbf{m} - \mathbf{e}_1) + (\lambda_0 + \lambda_2)V_{n-1}(\mathbf{m}), \\ 1 - \lambda_0 + (\lambda_1 + \lambda_2)V_{n-1}(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_{n-1}(\mathbf{m}), \\ \lambda_2 + \lambda_2 V_{n-1}(\mathbf{m} - \mathbf{e}_3) + (\lambda_0 + \lambda_1)V_{n-1}(\mathbf{m}) \end{array} \right\}, \quad (49)$$

where the seven terms in the max operator correspond to the action of offering slot types $\{1, 2, 3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1\}$, $\{2\}$ and $\{3\}$, respectively.

For ease of notation we define $\Delta_n^{ij}(\mathbf{m})$ to be the difference of the i th and j th terms in the max operator (49) above, $i, j \in \{1, 2, \dots, 7\}$. To prove the desired result it suffices to show for any $n \in \mathbb{N}$ that $\Delta_n^{3,j} \geq 0$, $j \neq 3$.

First, by writing out the definition,

$$\Delta_n^{35}(\mathbf{m}) = \lambda_2[1 + (V_{n-1}(\mathbf{m} - \mathbf{e}_3) - V_{n-1}(\mathbf{m}))] \geq 0, \quad (50)$$

$$\Delta_n^{37}(\mathbf{m}) = \lambda_1[1 + (V_{n-1}(\mathbf{m} - \mathbf{e}_1) - V_{n-1}(\mathbf{m}))] \geq 0. \quad (51)$$

The equalities follow from the fact that $V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m} - \mathbf{e}_3) \leq 1$ (for (50)) and $V_{n-1}(\mathbf{m}) - V_{n-1}(\mathbf{m} - \mathbf{e}_1) \leq 1$ (for (51)), see Lemma 3.(i).

The other four inequalities can be written as

$$\Delta_{n+1}^{31} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\lambda_1 + \lambda_2)V_n(\mathbf{m} - \mathbf{e}_2), \quad (52)$$

$$\Delta_{n+1}^{32} \geq 0 \Leftrightarrow \frac{1}{2}\lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\frac{1}{2}\lambda_1 + \lambda_2)V_n(\mathbf{m} - \mathbf{e}_2), \quad (53)$$

$$\Delta_{n+1}^{34} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \frac{1}{2}\lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\lambda_1 + \frac{1}{2}\lambda_2)V_n(\mathbf{m} - \mathbf{e}_2), \quad (54)$$

$$\Delta_{n+1}^{36} \geq 0 \Leftrightarrow \lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\lambda_1 + \lambda_2)V_n(\mathbf{m} - \mathbf{e}_2). \quad (55)$$

Note that (52) and (55) are equivalent, as are (53) and (54), due to symmetry. Thus, we limit ourselves to showing that (52) and (53) hold, which we will do by induction.

Let $n = 1$, then it is readily seen that for (52),

$$\lambda_1 V_1(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_1(\mathbf{m} - \mathbf{e}_3) = (\lambda_1 + \lambda_2)(1 - \lambda_0) = (\lambda_1 + \lambda_2)V_1(\mathbf{m} - \mathbf{e}_2), \quad (56)$$

and for (53),

$$\frac{1}{2}\lambda_1 V_1(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_1(\mathbf{m} - \mathbf{e}_3) = (\frac{1}{2}\lambda_1 + \lambda_2)(1 - \lambda_0) = (\frac{1}{2}\lambda_1 + \lambda_2)V_1(\mathbf{m} - \mathbf{e}_2), \quad (57)$$

so both hold.

Next we let $t \in \mathbb{N}$ and assume that (52)-(55) hold for all $n \leq t - 1$, i.e.,

$$\lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\lambda_1 + \lambda_2)V_n(\mathbf{m} - \mathbf{e}_2), \quad n \leq t - 1, \quad (58)$$

$$\frac{1}{2}\lambda_1 V_n(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_n(\mathbf{m} - \mathbf{e}_3) \geq (\frac{1}{2}\lambda_1 + \lambda_2)V_n(\mathbf{m} - \mathbf{e}_2), \quad n \leq t - 1. \quad (59)$$

In this case we know that g_n in (4) provides an optimal policy for all $n \leq t$. We shall now demonstrate that (58) and (59) hold for $n = t$ as well, which implies that g_n is also optimal for $n = t + 1$. Since we know an optimal control policy for $n \leq t$, we also know the transition probabilities given that we use optimal control.

$$p_0(\mathbf{m}) = \lambda_0 + \lambda_1 \mathbb{1}_{\{m_1=m_2=0\}} + \lambda_2 \mathbb{1}_{\{m_2=m_3=0\}}, \quad (60)$$

$$p_1(\mathbf{m}) = \lambda_1 \mathbb{1}_{\{m_1 \geq 1\}}, \quad (61)$$

$$p_2(\mathbf{m}) = \lambda_1 \mathbb{1}_{\{m_1=0, m_2 \geq 1\}} + \lambda_2 \mathbb{1}_{\{m_2 \geq 1, m_3=0\}}, \quad (62)$$

$$p_3(\mathbf{m}) = \lambda_2 \mathbb{1}_{\{m_3 \geq 1\}}. \quad (63)$$

Using the above transition probabilities we can compute

$$V_t(\mathbf{m} - \mathbf{e}_1) = 1 - \lambda_0 + \lambda_1 V_{t-1}(\mathbf{m} - 2\mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_1), \quad (64)$$

$$V_t(\mathbf{m} - \mathbf{e}_3) = 1 - \lambda_0 + \lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m} - 2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_3). \quad (65)$$

Moreover, we know from the induction hypothesis (58) that

$$\lambda_1 V_{t-1}(\mathbf{m} - 2\mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) \geq (\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2), \quad (66)$$

$$\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m} - 2\mathbf{e}_3) \geq (\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3). \quad (67)$$

Using (64)-(67), we can write

$$\begin{aligned} & \lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \\ & \geq (\lambda_1 + \lambda_2)(1 - \lambda_0) + \lambda_1(\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_2(\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3) \\ & \quad + \lambda_0(\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_3)) \\ & \geq (\lambda_1 + \lambda_2)(1 - \lambda_0) + \lambda_1(\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_2(\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3) \\ & \quad + \lambda_0(\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_2) \\ & = (\lambda_1 + \lambda_2) V_t(\mathbf{m} - \mathbf{e}_2), \end{aligned} \quad (68)$$

where the second inequality follows from the induction hypothesis (58). This proves the desired inequality.

Similarly, to verify (53) we use (64) and (65) and apply the induction hypothesis (59) to obtain, after some rearranging,

$$\begin{aligned} & \frac{1}{2} \lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \\ & \geq \left(\frac{1}{2} \lambda_1 + \lambda_2\right)(1 - \lambda_0) + \left(\frac{1}{2} \lambda_1 + \lambda_2\right) \lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \left(\frac{1}{2} \lambda_1 + \lambda_2\right) \lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3) \\ & \quad + \left(\frac{1}{2} \lambda_1 + \lambda_2\right) \lambda_0(\mathbf{m} - \mathbf{e}_2) \\ & = \left(\frac{1}{2} \lambda_1 + \lambda_2\right) V_t(\mathbf{m} - \mathbf{e}_2). \end{aligned} \quad (69)$$

$$= \left(\frac{1}{2} \lambda_1 + \lambda_2\right) V_t(\mathbf{m} - \mathbf{e}_2). \quad (70)$$

Next, we verify the induction hypotheses for the various boundary cases. First, it is readily verified, using our knowledge of the optimal control for $n = t$, that for $m_1 = 1$

$$\begin{aligned} V_t(\mathbf{m} - \mathbf{e}_1) &= (1 - \lambda_0) + \left(\lambda_1 + \frac{1}{2} \lambda_2\right) V_{t-1}(0, m_2 - 1, m_3) + \frac{1}{2} \lambda_2 V_{t-1}(0, m_2, m_3 - 1) + \lambda_0 V_{t-1}(0, m_2, m_3) \\ &\geq 1 - \lambda_0 + (\lambda_1 + \lambda_2) V_{t-1}(0, m_2 - 1, m_3) + \lambda_0 V_{t-1}(0, m_2, m_3), \end{aligned} \quad (71)$$

where the inequality follows from Lemma 4. Analogously, we derive

$$V_t(\mathbf{m} - \mathbf{e}_3) \geq 1 - \lambda_0 + (\lambda_1 + \lambda_2) V_{t-1}(m_1, m_2 - 1, 0) + \lambda_0 V_{t-1}(m_1, m_2, 0), \quad m_3 = 1. \quad (72)$$

First we treat the case $m_1 = 1$ and $m_3 \geq 2$. Combining (65) and (71) yields

$$\begin{aligned} & \lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \\ & \geq \lambda_1[(1 - \lambda_0) + (\lambda_1 + \lambda_2) V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_1)] \\ & \quad + \lambda_2[(1 - \lambda_0) + \lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m} - 2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_3)] \\ & \geq (\lambda_1 + \lambda_2)(1 - \lambda_0) + (\lambda_1 + \lambda_2) \lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + (\lambda_1 + \lambda_2) \lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3) \\ & \quad + (\lambda_1 + \lambda_2) \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_2) \\ & = (\lambda_1 + \lambda_2) V_t(\mathbf{m} - \mathbf{e}_2), \end{aligned} \quad (73)$$

with the second inequality due to the induction hypothesis (58).

In order to show (59) we can again use (65) and (71), and do some rearranging to show that

$$\begin{aligned}
& \frac{1}{2}\lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \\
& \geq \frac{1}{2}\lambda_1 [(1 - \lambda_0) + (\lambda_1 + \lambda_2)V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_1)] \\
& \quad + \lambda_2 [(1 - \lambda_0) + \lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) + \lambda_2 V_{t-1}(\mathbf{m} - 2\mathbf{e}_3) + \lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_3)] \\
& \geq (\frac{1}{2}\lambda_1 + \lambda_2)(1 - \lambda_0) + (\lambda_1 + \lambda_2)\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_2[\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_3) \\
& \quad + (\frac{1}{2}\lambda_1 + \lambda_2)V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3)] + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_2) \tag{74}
\end{aligned}$$

$$\begin{aligned}
& \geq (\frac{1}{2}\lambda_1 + \lambda_2)(1 - \lambda_0) + (\lambda_1 + \lambda_2)\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) + \lambda_2[\frac{1}{2}\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) \\
& \quad + (\frac{1}{2}\lambda_1 + \lambda_2)V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3)] + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_2) \tag{75}
\end{aligned}$$

$$= (\frac{1}{2}\lambda_1 + \lambda_2)V_{t-1}(\mathbf{m} - \mathbf{e}_2), \tag{76}$$

where the second and third equalities follows from the induction hypothesis (59) and Lemma 4, respectively. This shows that the (59) holds for $m_1 = 1, m_3 \geq 2$.

The proof for the case $m_1 \geq 2, m_3 = 1$ follows from symmetry. Finally, we verify the case $m_1 = m_3 = 1$. We first bound, using (71) and (72),

$$\begin{aligned}
& \lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \geq (\lambda_1 + \lambda_2)(1 - \lambda_0) + (\lambda_1 + \lambda_2)\lambda_1 V_{t-1}(\mathbf{m} - \mathbf{e}_1 - \mathbf{e}_2) \\
& \quad + (\lambda_1 + \lambda_2)\lambda_2 V_{t-1}(\mathbf{m} - \mathbf{e}_2 - \mathbf{e}_3) + (\lambda_1 + \lambda_2)\lambda_0 V_{t-1}(\mathbf{m} - \mathbf{e}_2) \\
& = (\lambda_1 + \lambda_2)V_t(\mathbf{m} - \mathbf{e}_2). \tag{77}
\end{aligned}$$

Using these same inequalities we can show

$$\begin{aligned}
& \frac{1}{2}\lambda_1 V_t(\mathbf{m} - \mathbf{e}_1) + \lambda_2 V_t(\mathbf{m} - \mathbf{e}_3) \\
& \geq \frac{1}{2}\lambda_1 [(1 - \lambda_0) + (\lambda_1 + \lambda_2)V_{t-1}(0, m_2 - 1, 1) + \lambda_0 V_{t-1}(0, m_2, 1)] \\
& \quad + \lambda_2 [(1 - \lambda_0) + (\lambda_2 + \frac{1}{2}\lambda_1)V_{t-1}(1, m_2 - 1, 0) + \frac{1}{2}\lambda_1 V_{t-1}(0, m_2, 0) + \lambda_0 V_{t-1}(1, m_2, 0)] \\
& \geq (\frac{1}{2}\lambda_1 + \lambda_2)(1 - \lambda_0) + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_2 V_{t-1}(1, m_2 - 1, 0) + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_1 V_{t-1}(0, m_2 - 1, 1) \\
& \quad + \lambda_0[\frac{1}{2}\lambda_1 V_{t-1}(0, m_2, 1) + \lambda_2 V_{t-1}(1, m_2, 0)] \\
& \geq (\frac{1}{2}\lambda_1 + \lambda_2)(1 - \lambda_0) + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_2 V_{t-1}(1, m_2 - 1, 0) + (\frac{1}{2}\lambda_1 + \lambda_2)\lambda_1 V_{t-1}(0, m_2 - 1, 1) \\
& \quad + \lambda_0(\frac{1}{2}\lambda_1 + \lambda_2)V_{t-1}(1, m_2 - 1, 1) \\
& = (\frac{1}{2}\lambda_1 + \lambda_2)V_t(\mathbf{m} - \mathbf{e}_2), \tag{78}
\end{aligned}$$

with the second inequality using Lemma 4.(i). and the third inequality due to the induction hypothesis (59). This completes the proof. \square \square

F Proof of Corollary 1

We prove by contradiction. Suppose (5) does not hold and thus

$$V_n(\mathbf{m} - \mathbf{e}_2) > V_n(\mathbf{m} - \mathbf{e}_1) \text{ and } V_n(\mathbf{m} - \mathbf{e}_2) > V_n(\mathbf{m} - \mathbf{e}_3). \tag{79}$$

In period $n + 1$ and at state \mathbf{m} , action $\mathbf{d}_1 := (1, 0, 1)$ yields the value-to-go of

$$p_1(\mathbf{m}, \mathbf{d}_1)V_n(\mathbf{m} - \mathbf{e}_1) + p_3(\mathbf{m}, \mathbf{d}_1)V_n(\mathbf{m} - \mathbf{e}_3) + \lambda_0 V_n(\mathbf{m}), \quad (80)$$

which is strictly less than the value-to-go under action $\mathbf{d}_2 = (0, 1, 0)$ given by

$$p_2(\mathbf{m}, \mathbf{d}_2)V_n(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_n(\mathbf{m}), \quad (81)$$

by using (79) and $p_1(\mathbf{m}, \mathbf{d}_1) + p_3(\mathbf{m}, \mathbf{d}_1) = p_2(\mathbf{m}, \mathbf{d}_2) = 1 - \lambda_0$. This contradicts the result in Proposition 2 on the optimality of \mathbf{d}_1 . \square

G Proof of Lemma 1

Proof. Proof. The proof follows that of Lemma 3 with some minor modifications. In particular, to prove (8), we define a decision rule \mathbf{h} in period $t + 1$ which acts the same as $\mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j)$ regarding all slot types but type j . For type j , \mathbf{h} does not offer it in any subsets it offers. That is, $\mathbf{h} = \mathbf{g}_{t+1}^*(\mathbf{m} + \mathbf{e}_j)$ except that we enforce $h_{kj} = 0, \forall k$. All other parts of the proof readily follow. \square \square

H Proof of Lemma 2

Proof. Proof. For convenience, we introduce some notation here. We say the decision rule in period n given the system occupies state $\mathbf{m} \in S$ can be described by a matrix-valued function: $\mathbf{g}_n : S \rightarrow \mathbf{d}$ in which $\mathbf{d} = \{d_{kj}\}$ is a J by J matrix, $d_{kj} \in \{0, 1\}$. If $d_{kj} = 1$, type j slots are offered in the k th subset. Since these subsets offered are mutually exclusive, $\sum_{k=1}^K d_{kj} \leq 1, \forall j$. As before, depleted slot types cannot be offered: $d_{kj} \leq m_j$.

Let $\hat{\mathbf{d}}$ denote an optimal decision rule. Without loss of generality, we assume that $m_j > 0, \forall j \in \mathcal{J}$. Otherwise we would consider a network where the preference matrix has been modified by removing empty slots. Let $\hat{\mathcal{J}} = \{j : \sum_{k=1}^{K-1} d_{kj} = 1, j \in \mathcal{J}\}$ be the set of slot types offered by $\hat{\mathbf{d}}$ collectively in all subsets it offers. Assume that $\mathcal{J} \setminus \hat{\mathcal{J}} \neq \emptyset$. Consider another decision rule $\tilde{\mathbf{d}}$ which follows exactly the same sequential offering rule as $\hat{\mathbf{d}}$, and then offers all slots types in $\mathcal{J} \setminus \hat{\mathcal{J}}$ as the K th offer set. So $\tilde{\mathbf{d}}$ eventually offers all slot types. To prove the desired result, it suffices to show that $\tilde{\mathbf{d}}$ is no worse than $\hat{\mathbf{d}}$, and thus must be optimal as well.

First consider a policy that uses $\hat{\mathbf{d}}$ in the first slot, and then follows the optimal scheduling rule. Let $V_n^{\hat{\mathbf{d}}}(\mathbf{m})$ denote the expected objective value following such a policy. Recall that $V_n(\mathbf{m})$ is the optimal expected objective value, and that $p_j(\mathbf{m}, \mathbf{d})$ denotes the probability that a type j slot will be booked in state \mathbf{m} if decision rule \mathbf{d} is used. It follows that

$$V_n^{\hat{\mathbf{d}}}(\mathbf{m}) = \sum_{j \in \hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}}) + \sum_{j \in \hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}}) V_{n-1}(\mathbf{m} - \mathbf{e}_j) + [1 - \sum_{j \in \hat{\mathcal{J}}} p_j(\mathbf{m}, \hat{\mathbf{d}})] V_{n-1}(\mathbf{m}). \quad (82)$$

Then, consider a policy that uses $\tilde{\mathbf{d}}$ first, and then follow the optimal scheduling rule. The expected objective valuing of this policy is

$$V_n^{\tilde{\mathbf{d}}}(\mathbf{m}) = \sum_{j \in \mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}}) + \sum_{j \in \mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}}) V_{n-1}(\mathbf{m} - \mathbf{e}_j) + [1 - \sum_{j \in \mathcal{J}} p_j(\mathbf{m}, \tilde{\mathbf{d}})] V_{n-1}(\mathbf{m}) \quad (83)$$

It is easy to check that $p_j(\mathbf{m}, \hat{\mathbf{d}}) = p_j(\mathbf{m}, \tilde{\mathbf{d}})$ for $j \in \hat{\mathcal{J}}$, as $\hat{\mathbf{d}}$ acts the same as $\tilde{\mathbf{d}}$ in the first $K - 1$ offer sets that cover slots types in $\hat{\mathcal{J}}$. Subtracting (82) from (83) and simplifying, we arrive at

$$V_n^{\tilde{\mathbf{d}}}(\mathbf{m}) - V_n^{\hat{\mathbf{d}}}(\mathbf{m}) = \sum_{j \in \mathcal{J} \setminus \hat{\mathcal{J}}} p_j(\mathbf{m}, \tilde{\mathbf{d}}) (1 + V_{n-1}(\mathbf{m} - \mathbf{e}_j) - V_{n-1}(\mathbf{m})) \geq 0,$$

where the last inequality directly follows from Lemma 1, proving the desired result. \square \square

I Proof of Theorem 3

Proof. Proof. Lemma 2 suggests that there exists an optimal decision rule $\mathbf{S}^* = S_1^* \dots S_K^*$ such that $\cup_{i=1}^K S_i^* = J$. Suppose that $S_1^* \dots S_K^*$ does not take the form as desired, we will show below that the objective value obtained by partitioning \mathbf{S}^* into singletons $\{j_1\} \dots \{j_J\}$ is no worse than that of $S_1^* \dots S_K^*$.

If there exists some k that $|S_k^*| > 1$, let us consider an alternative decision rule

$$\hat{\mathbf{S}}^* = S_1^* - \dots - S_{k-1}^* - \{t_1\} - S_k^* \setminus \{t_1\} - \dots - S_K^*, \quad (84)$$

such that

$$V_{n-1}(\mathbf{m} - \mathbf{e}_{t_1}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_t), \quad \forall t \in S_k^* \setminus \{t_1\}. \quad (85)$$

This new decision rule follows the same offering sequence as the original rule, except that it splits the offer set S_k^* into two sub-offer sets S_{k-1}^* and $\{t_1\}$.

Now, we will show that $\hat{\mathbf{S}}^*$ does no worse than \mathbf{S}^* . To do that, let $V_n^1(\mathbf{m})$ be the expected number of slots filled at the end of the booking horizon by following decision rule $\hat{\mathbf{S}}^*$ at period n and then following the optimal decision afterwards. Let $\Delta^1 = V_n(\mathbf{m}) - V_n^1(\mathbf{m})$. Let $I^* = \{i : \Omega_{it_1} = 1, \sum_{j \in S_k^* \setminus \{t_1\}} \Omega_{ij} \geq 1, \sum_{j \in \cup_{i=1}^{k-1} S_i^*} \Omega_{ij} = 0\}$ be the set of patient types that accept type t_1 slots and also at least one slot type in the set of $S_k^* \setminus \{t_1\}$, but do not accept any slot type that has been offered so far in sets S_1^* through S_{k-1}^* . Let $J^*(i) = \{j : j \in S_k^*, \Omega_{ij} = 1\}$ be the subset of slots type in S_k^* that are acceptable by patient type i , $i \in I^*$. Clearly, $t_1 \in J^*(i)$. One can find that

$$\Delta^1 = \sum_{i \in I^*} \frac{\lambda_i}{|J^*(i)|} \sum_{j \in J^*(i)} V_{n-1}(\mathbf{m} - \mathbf{e}_j) - \sum_{i \in I^*} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{t_1}) \leq 0,$$

where the last equality follows from (85), proving that $\hat{\mathbf{S}}^*$ does no worse than \mathbf{S}^* .

Following the procedure above to keep splitting offer sets that contain more than one slot types, we can obtain an optimal action of form $\{j'_1\} \dots \{j'_J\}$ so that each sequential offer set contains exactly one slot type. Suppose that $\{j'_1\} \dots \{j'_J\}$ does not follow the order desired. That is, there exists $1 \leq u \leq J+1$ such that $V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_u}) < V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_{u+1}})$. Consider another decision rule with only j'_u and j'_{u+1} switched and others remained the same order.

$$\{j'_1\} - \dots - \{j'_{u+1}\} - \{j'_u\} - \dots - \{j'_J\}. \quad (86)$$

It suffices to show the claim that (86) either provides the same objective value as $\{j'_1\} \dots \{j'_J\}$, or strictly higher, which contradicts with the optimality of $\{j'_1\} \dots \{j'_J\}$, and thus for all $1 \leq u \leq J+1$, $V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_u}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_{u+1}})$ as desired.

To show the claim above, let $I' = \{i : \Omega_{ij'_u} = 1, \Omega_{ij'_{u+1}} = 1, \sum_{v=1}^{u-1} \Omega_{ij'_v} = 0\}$ be the set of patient types that accept both types j'_u and j'_{u+1} slots, but do not accept any slot type that has been offered so far. Let $V_n^2(\mathbf{m})$ be the expected number of slots filled at the end of the booking horizon by following decision rule (86) at period n and then following the optimal decision afterwards. We consider

$$\Delta^2 = V_n(\mathbf{m}) - V_n^2(\mathbf{m}) = \sum_{i \in I'} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_u}) - \sum_{i \in I'} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j'_{u+1}}).$$

If $\sum_{i \in I'} \lambda_i = 0$, then $\Delta^2 = 0$ and thus (86) is optimal. However, if $\sum_{i \in I'} \lambda_i > 0$, then $\Delta^2 < 0$ leading to the contradiction desired. This proves our claim and completes the proof. \square \square

J Proof of Theorem 4

Proof. Proof. For notational convenience, here we consider the case when $\lambda_0 = 0$ and all patient types $i \in \mathcal{I}$ can be covered by at least one slot type left in \mathbf{m} . Proofs of other cases follow a similar procedure.

It is trivial that $V_n^s(\mathbf{m}) = V_n^f(\mathbf{m})$, for $n = 0, 1$ and for all $\mathbf{m} \geq 0$. Assume the desired equality holds up to $n = t - 1$, and consider $n = t$. Let $V_n^f(\mathbf{m}|i)$ be the optimal value function with system state \mathbf{m} , the current arrival being patient type $i \in \mathcal{I}$ and n periods to go. Then $\forall i \in \mathcal{I}$,

$$V_n^f(\mathbf{m}|i) = \max_{\mathbf{d}} \left\{ \sum_{j=1}^J p_{ij}(\mathbf{m}, \mathbf{d}) [1 + V_{n-1}^f(\mathbf{m} - \mathbf{e}_j)] \right\},$$

where \mathbf{d} is the offered set and $p_{ij}(\mathbf{m}, \mathbf{d})$ is the probability that slot type j will be taken if \mathbf{d} is offered and the arrival is type i patient. It is not difficult to see that the optimal offer set will be the slot type $j^*(i)$ (which is a function of i) such that

$$j^*(i) = \arg \max_{j \in \{k: \Omega_{ik}=1, k \in \mathcal{J}\}} V_{n-1}^f(\mathbf{m} - \mathbf{e}_j). \quad (87)$$

That is, for any arriving patient type, the optimal action is to offer the slot type that is acceptable by this patient type and that leads to the largest value-to-go. It follows that

$$V_n^f(\mathbf{m}) = \sum_{i \in \mathcal{I}} \lambda_i V_n^f(\mathbf{m}|i) = 1 + \sum_{i \in \mathcal{I}} \lambda_i V_{n-1}^f(\mathbf{m} - \mathbf{e}_{j^*(i)}) = 1 + \sum_{i \in \mathcal{I}} \lambda_i V_{n-1}^s(\mathbf{m} - \mathbf{e}_{j^*(i)}) = V_n^s(\mathbf{m}).$$

To see the last equality, note that the optimal action stipulated by Theorem 3 ensures that (i) for any arriving patient type, it will find an acceptable slot type and (ii) this accepted slot type leads to the largest value-to-go among all slot types accepted by this patient type. This is exactly enforced by (87). \square

K Proof of Theorem 5

Proof. Proof. For notational convenience, here we consider the case when $\lambda_0 = 0$. The proof for the case when $\lambda_0 > 0$ follows similar steps. In light of Theorem 3, it suffices to show that for any j_1, j_2 such that $I(j_1) \subset I(j_2)$,

$$V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1}) \geq V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2}), \forall n = 1, 2, \dots, N. \quad (88)$$

It is easy to see that (88) holds for $n = 1$. Assume it holds up to $n > 1$ and consider $n + 1$. Let us consider a few cases below.

Case 1: $\mathbf{m}_{j_1} \geq 2, \mathbf{m}_{j_2} \geq 1$. Let g^* be an optimal action for the state $\mathbf{m} - \mathbf{e}_{j_2}$. Let $B_{ij}^g(\mathbf{m})$ denote the probability that a type i patients will choose type j slots in state \mathbf{m} when action g is taken. If $j = 0$, then no slots are chosen. Note that g^* is always feasible for state $\mathbf{m} - \mathbf{e}_{j_1}$, and that $B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_1}) = B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2})$. Thus,

$$\begin{aligned} V_n(\mathbf{m} - \mathbf{e}_{j_1}) &\geq V_n^{g^*}(\mathbf{m} - \mathbf{e}_{j_1}) = \sum_{i=1}^I \lambda_i \sum_{j=0}^J B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_1}) [\mathbb{1}_{j>0} + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_j)] \\ &\geq \sum_{i=1}^I \lambda_i \sum_{j=0}^J B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) [\mathbb{1}_{j>0} + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_j)] = V_n^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) = V_n(\mathbf{m} - \mathbf{e}_{j_2}), \end{aligned}$$

where the second inequality follows from the induction hypothesis.

Case 2: $\mathbf{m}_{j_1} = 1, \mathbf{m}_{j_2} \geq 2$. Again, let g^* be an optimal action for the state $\mathbf{m} - \mathbf{e}_{j_2}$. Following induction hypothesis, we choose g^* so that a slot type with a smaller set of covered patient types will be offered before any other slot type with a larger covered set of patient types. Thus, g^* offers j_1 before offering j_2 . Let \tilde{g} be an action that follows exactly as g^* except that \tilde{g} does not offer type j_1 slots. It is clear that \tilde{g} is feasible for state $\mathbf{m} - \mathbf{e}_{j_1}$. There are two subcases.

Case 2a: None of the slot types if any offered between j_1 and j_2 by g^* are acceptable by patient type i_1 , $\forall i_1 \in \underline{I}(j_1)$ where $\underline{I}(j_1)$ represent the set of patient types who would choose slot type j_1 when it is offered by g^* at state $\mathbf{m} - \mathbf{e}_{j_2}$. The following inequalities hold.

$$\begin{aligned} B_{ij}^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) &= B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}), \forall i \notin \underline{I}(j_1), \forall j; \\ B_{i j_1}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) &= B_{i j_2}^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) = 1, \forall i \in \underline{I}(j_1). \end{aligned}$$

It follows that

$$\begin{aligned}
V_n(\mathbf{m} - \mathbf{e}_{j_1}) &\geq V_n^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) = \sum_{i=1}^I \lambda_i \sum_{j=1}^J B_{ij}^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_j)] \\
&= \sum_{i \notin \underline{I}(j_1)} \lambda_i \sum_{j=1}^J B_{ij}^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_j)] + \sum_{i \in \underline{I}(j_1)} \lambda_i B_{ij_2}^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j_2})] \\
&\geq \sum_{i \notin \underline{I}(j_1)} \lambda_i \sum_{j=1}^J B_{ij}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_j)] + \sum_{i \in \underline{I}(j_1)} \lambda_i B_{ij_1}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j_1})] \\
&= V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2}),
\end{aligned}$$

where the last inequality follows from the induction hypothesis.

Case 2b: Let $\underline{I}(j_k)$ be the set of the patient types who will actually choose slot type j_k under g^* when it is offered at state $\mathbf{m} - \mathbf{e}_{j_2}$. One slot type, say j_3 , offered between j_1 and j_2 by g^* is acceptable by some patient type $i_1 \in \underline{I}(j_1)$. Consider j_3 to be the only one of such slots (extension to multiple of such slots uses a similar but more tedious proof). Because slot types j_1 , j_2 and j_3 can cover some same patient types, we know that either $\underline{I}(j_3) \subset \underline{I}(j_k)$ or $\underline{I}(j_3) \supset \underline{I}(j_k)$, $k = 1, 2$, by the preassumption of the theorem. Because we choose g^* based on the size of covered set of patient types, we have that $\underline{I}(j_1) \subset \underline{I}(j_2) \subset \underline{I}(j_3)$. Also note that $\cap_{k=1}^3 \underline{I}(j_k) = \emptyset$ because any patient type can choose at most one slot type. Let $j(i)$ be the slot type chosen by patient type i , $i \notin \underline{I}(j_1) \cup \underline{I}(j_2) \cup \underline{I}(j_3)$. Note that $j(i)$ is the same under g^* or \tilde{g} , $\forall i \notin \underline{I}(j_1) \cup \underline{I}(j_2) \cup \underline{I}(j_3)$. It follows that

$$\begin{aligned}
V_n(\mathbf{m} - \mathbf{e}_{j_1}) - V_n(\mathbf{m} - \mathbf{e}_{j_2}) &\geq V_n^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) - V_{n-1}^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) \\
&= \sum_{i \notin \underline{I}(j_1) \cup \underline{I}(j_2) \cup \underline{I}(j_3)} \lambda_i [V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j(i)}) - V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j(i)})] \\
&\quad + \sum_{i \in \underline{I}(j_1) \cup \underline{I}(j_3)} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j_3}) + \sum_{i \in \underline{I}(j_2)} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j_2}) \\
&\quad - \{ \sum_{i \in \underline{I}(j_1)} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j_1}) + \sum_{i \in \underline{I}(j_3)} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j_3}) + \sum_{i \in \underline{I}(j_2)} \lambda_i V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j_2}) \} \geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis.

Case 3: $\mathbf{m}_{j_1} = 1, \mathbf{m}_{j_2} = 1$. Let g^* be an optimal action for the state $\mathbf{m} - \mathbf{e}_{j_2}$ (g^* does not offer slot type j_2 because none is available). Let \tilde{g} be an action that follows exactly as g^* except that \tilde{g} does not offer type j_1 slots but offers type j_2 at the end. It is clear that \tilde{g} is feasible for state $\mathbf{m} - \mathbf{e}_{j_1}$. Let $\underline{I}(j_2)$ be the patient types who choose j_2 under \tilde{g} ; these patients do not book any appointments under g^* . Let $j(i)$ be the slot type actually chosen by patient type i , $i \notin \underline{I}(j_2)$. Note that $j(i)$ is the same under g^* or \tilde{g} . It follows that

$$\begin{aligned}
V_n(\mathbf{m} - \mathbf{e}_{j_1}) - V_n(\mathbf{m} - \mathbf{e}_{j_2}) &\geq V_n^{\tilde{g}}(\mathbf{m} - \mathbf{e}_{j_1}) - V_n^{g^*}(\mathbf{m} - \mathbf{e}_{j_2}) \\
&= \sum_{i \notin \underline{I}(j_2)} \lambda_i [V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j(i)}) + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2} - \mathbf{e}_{j(i)})] + \sum_{i \in \underline{I}(j_2)} \lambda_i [1 + V_{n-1}(\mathbf{m} - \mathbf{e}_{j_1} - \mathbf{e}_{j_2}) - V_{n-1}(\mathbf{m} - \mathbf{e}_{j_2})] \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the induction hypothesis and Lemma 1. This completes the whole proof. \square

L Proof of Proposition 3

Proof. Proof. It suffices to show the following monotonic results for the “W” model with sequential offers: $V_n(\mathbf{m} - \mathbf{e}_1) - V_n(\mathbf{m} - \mathbf{e}_2)$ increases as m_1 increases and that $V_n(\mathbf{m} - \mathbf{e}_2) - V_n(\mathbf{m} - \mathbf{e}_1)$ increases as m_2 increases, $\forall n \geq 1, \forall \mathbf{m} \geq (1, 1)$.

That is,

$$V_n(\mathbf{m}) - V_n(\mathbf{m} + \mathbf{e}_1 - \mathbf{e}_2) \geq V_n(\mathbf{m} - \mathbf{e}_1) - V_n(\mathbf{m} - \mathbf{e}_2), \quad \forall n \geq 1, \quad \forall \mathbf{m} \geq (1, 1), \quad (89)$$

and

$$V_n(\mathbf{m}) - V_n(\mathbf{m} + \mathbf{e}_2 - \mathbf{e}_1) \geq V_n(\mathbf{m} - \mathbf{e}_2) - V_n(\mathbf{m} - \mathbf{e}_1), \quad \forall n \geq 1, \quad \forall \mathbf{m} \geq (1, 1). \quad (90)$$

To facilitate the proof of (89) and (90), we introduce a few notations. Let $\Delta_n^A(\mathbf{m}) = V_n(\mathbf{m} - \mathbf{e}_1) - V_n(\mathbf{m} - \mathbf{e}_2)$ and $\Delta_n^B(\mathbf{m}) = V_n(\mathbf{m} - \mathbf{e}_2) - V_n(\mathbf{m} - \mathbf{e}_1)$. Note that (89) and (90) are symmetric, and thus we limit ourselves to just prove (89).

Consider the case for $n = 1$. At $\mathbf{m} = (1, 1)$, $\Delta_1^A(\mathbf{m}) = V_1(0, 1) - V_1(1, 0) = (\lambda_2 + \lambda_3) - (\lambda_1 + \lambda_2) = \lambda_3 - \lambda_1$. For $\mathbf{m} = (m_1, 1)$ and $m_1 \geq 2$, we have $\Delta_1^A(\mathbf{m}) = V_1(m_1 - 1, 1) - V_1(m_1, 0) = (1 - \lambda_0) - (\lambda_1 + \lambda_2) = \lambda_3$. Thus, (89) holds for $n = 1$, $\mathbf{m} = (m_1, 1)$ and $m_1 \geq 1$. Now, for $n = 1$ and $\mathbf{m} = (1, m_2)$ and $m_2 \geq 2$, we have $\Delta_1^A(\mathbf{m}) = V_1(0, m_2) - V_1(1, m_2 - 1) = (\lambda_2 + \lambda_3) - (1 - \lambda_0) = -\lambda_1$. Consider $n = 1$, $\mathbf{m} = (m_1, m_2)$, and $m_1, m_2 \geq 2$. In this case, we have that $\Delta_1^A(\mathbf{m}) = V_1(m_1 - 1, m_2) - V_1(m_1, m_2 - 1) = (1 - \lambda_0) - (1 - \lambda_0) = 0$. Thus, (89) holds for $n = 1$, $\mathbf{m} = (m_1, m_2)$ and $m_1 \geq 1, m_2 \geq 2$. This completes the proof of (89) for $n = 1$.

Assume that (89) holds up to $n = k$ for $\mathbf{m} \geq (1, 1)$. We will use induction below to show that this is also true for $n = k + 1$. We start by writing the Bellman's equation below.

$$V_{k+1}(\mathbf{m}) = \max \left\{ \begin{array}{l} 1 - \lambda_0 + (\lambda_1 + \frac{1}{2}\lambda_2)V_k(\mathbf{m} - \mathbf{e}_1) + (\frac{1}{2}\lambda_2 + \lambda_3)V_k(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}), \\ 1 - \lambda_0 + (\lambda_1 + \lambda_2)V_k(\mathbf{m} - \mathbf{e}_1) + \lambda_3 V_k(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}), \\ 1 - \lambda_0 + \lambda_1 V_k(\mathbf{m} - \mathbf{e}_1) + (\lambda_2 + \lambda_3)V_k(\mathbf{m} - \mathbf{e}_2) + \lambda_0 V_k(\mathbf{m}). \end{array} \right\}, \quad (91)$$

where the three terms in the max operator correspond to actions $\{1, 2\}$, $\{1\}$ - $\{2\}$ and $\{2\}$ - $\{1\}$, respectively. Action $\{S_1\}$ - $\{S_2\}$ offers subset S_1 followed by subset S_2 . For ease of notation, we define $\Delta_{k+1}^{ij}(\mathbf{m})$ to be the difference of the i th and j th terms in the max operator (91) above, $i, j \in \{1, 2, 3\}$. It follows that

$$\Delta_{k+1}^{21}(\mathbf{m}) = \frac{1}{2}\lambda_2[V_k(\mathbf{m} - \mathbf{e}_1) - V_k(\mathbf{m} - \mathbf{e}_2)] = \frac{1}{2}\lambda_2\Delta_k^A(\mathbf{m}),$$

and

$$\Delta_{k+1}^{31}(\mathbf{m}) = \frac{1}{2}\lambda_2[V_k(\mathbf{m} - \mathbf{e}_2) - V_k(\mathbf{m} - \mathbf{e}_1)] = \frac{1}{2}\lambda_2\Delta_k^B(\mathbf{m}).$$

Because $\Delta_{k+1}^{21}(\mathbf{m}) + \Delta_{k+1}^{31}(\mathbf{m}) = 0$, one of these two terms must be non-negative suggesting one of the corresponding actions is optimal. In particular, if $\Delta_{k+1}^{21}(\mathbf{m}) \geq 0$, or equivalently, $\Delta_k^A(\mathbf{m}) \geq 0$, the the optimal action is $\{1\}$ - $\{2\}$; otherwise, it would be $\{2\}$ - $\{1\}$.

To prove the desired result, we need to consider the following cases. Case (1): $\mathbf{m} = (1, 1)$; case (2): $\mathbf{m} = (m_1, 1)$, $m_1 \geq 2$; case (3): $\mathbf{m} = (1, m_2)$, $m_2 \geq 2$; and case (4), $\mathbf{m} \geq (2, 2)$.

For Case (1) with $\mathbf{m} = (1, 1)$, we have

$$\Delta_{k+1}^A(1, 1) = V_{k+1}(0, 1) - V_{k+1}(1, 0) = [(1 - \lambda_1 - \lambda_0) + (\lambda_1 + \lambda_0)V_k(0, 1)] - [(1 - \lambda_3 - \lambda_0) + (\lambda_3 + \lambda_0)V_k(1, 0)].$$

We consider two subcases. Case (1a): if at state $(1, 1)$ the optimal action is $\{1\}$ - $\{2\}$, then

$$\begin{aligned} \Delta_{k+1}^A(2, 1) &= V_{k+1}(1, 1) - V_{k+1}(2, 0) \\ &= [(1 - \lambda_0) + (\lambda_1 + \lambda_2)V_k(0, 1) + \lambda_3 V_k(1, 0) + \lambda_0 V_k(1, 1)] \\ &\quad - [(1 - \lambda_3 - \lambda_0) + (\lambda_1 + \lambda_2)V_k(1, 0) + (\lambda_3 + \lambda_0)V_k(2, 0)]. \end{aligned}$$

It follows that

$$\begin{aligned} &\Delta_{k+1}^A(2, 1) - \Delta_{k+1}^A(1, 1) \\ &= \lambda_1[1 - V_k(1, 0)] + \lambda_2[V_k(0, 1) - V_k(1, 0)] + \lambda_3[2V_k(1, 0) - V_k(2, 0)] + \lambda_0[\Delta_k^A(2, 1) - \Delta_k^A(1, 1)] \\ &= \lambda_1[1 - V_k(1, 0)] + \lambda_2\Delta_k^A(1, 1) + \lambda_3[2V_k(1, 0) - V_k(2, 0)] + \lambda_0[\Delta_k^A(2, 1) - \Delta_k^A(1, 1)]. \end{aligned}$$

It is trivial that $1 - V_k(1, 0) \geq 0$. We also know that $\Delta_k^A(1, 1) \geq 0$ in this case because the optimal action is $\{1\}$ - $\{2\}$; and that $\Delta_k^A(2, 1) - \Delta_k^A(1, 1) \geq 0$ by the induction hypothesis. Finally, we claim that

$$2V_k(1, 0) - V_k(2, 0) \geq 0, \quad \forall k \geq 1, \quad (92)$$

which will be shown at the end of this proof. Thus, $\Delta_{k+1}^A(2, 1) - \Delta_{k+1}^A(1, 1) \geq 0$ if the optimal action is $\{1\}$ - $\{2\}$ at state $(1, 1)$.

Case (1b): if the optimal action at state $(m_1, 1)$ is $\{2\}$ - $\{1\}$, then

$$\begin{aligned} \Delta_{k+1}^A(2, 1) &= V_{k+1}(1, 1) - V_{k+1}(2, 0) \\ &= [(1 - \lambda_0) + \lambda_1 V_k(0, 1) + (\lambda_2 + \lambda_3) V_k(1, 0) + \lambda_0 V_k(1, 1)] \\ &\quad - [(1 - \lambda_3 - \lambda_0) + (\lambda_1 + \lambda_2) V_k(1, 0) + (\lambda_3 + \lambda_0) V_k(2, 0)]. \end{aligned}$$

It follows that

$$\begin{aligned} &\Delta_{k+1}^A(2, 1) - \Delta_{k+1}^A(1, 1) \\ &= \lambda_1 [1 - V_k(1, 0)] + \lambda_3 [2V_k(1, 0) - V_k(2, 0)] + \lambda_0 [\Delta_k^A(2, 1) - \Delta_k^A(1, 1)] \geq 0. \end{aligned}$$

In summary, cases (1a) and (1b) collectively show that $\Delta_{k+1}^A(2, 1) - \Delta_{k+1}^A(1, 1) \geq 0$.

For Case(2) with $\mathbf{m} = (m_1, 1)$, $m_1 \geq 2$, we evaluate $V_{k+1}(m_1 - 1, 1) - V_{k+1}(m_1, 0)$ in the following two subcases. Case(2a): if the optimal action at state $(m_1, 1)$ is $\{1\}$ - $\{2\}$, then

$$\begin{aligned} &V_{k+1}(m_1 - 1, 1) - V_{k+1}(m_1, 0) \\ &= [(1 - \lambda_0) + (\lambda_1 + \lambda_2) V_k(m_1 - 2, 1) + \lambda_3 V_k(m_1 - 1, 0) + \lambda_0 V_k(m_1 - 1, 1)] \\ &\quad - [(1 - \lambda_3 - \lambda_0) + (\lambda_1 + \lambda_2) V_k(m_1 - 1, 0) + (\lambda_3 + \lambda_0) V_k(m_1, 0)] \\ &= \lambda_3 + (\lambda_1 + \lambda_2) \Delta_k^A(m_1 - 1, 1) + \lambda_0 \Delta_k^A(m_1, 1) + \lambda_3 [(1 - (\lambda_1 + \lambda_2)^{m_1 - 1}) - (1 - (\lambda_1 + \lambda_2)^{m_1})] \\ &= \lambda_3 + (\lambda_1 + \lambda_2) \Delta_k^A(m_1 - 1, 1) + \lambda_0 \Delta_k^A(m_1, 1) - \lambda_3 (\lambda_3 + \lambda_0) (\lambda_1 + \lambda_2)^{m_1 - 1}, \end{aligned}$$

which increases as m_1 increases by the induction hypothesis.

Case(2b): if the optimal action at state $(m_1, 1)$ is $\{2\}$ - $\{1\}$, then for $\mathbf{m} = (m_1, 1)$, $m_1 \geq 2$,

$$\begin{aligned} &V_{k+1}(m_1 - 1, 1) - V_{k+1}(m_1, 0) \\ &= [(1 - \lambda_0) + \lambda_1 V_k(m_1 - 2, 1) + (\lambda_2 + \lambda_3) V_k(m_1 - 1, 0) + \lambda_0 V_k(m_1 - 1, 1)] \\ &\quad - [(1 - \lambda_3 - \lambda_0) + (\lambda_1 + \lambda_2) V_k(m_1 - 1, 0) + (\lambda_3 + \lambda_0) V_k(m_1, 0)] \\ &= \lambda_3 + \lambda_1 \Delta_k^A(m_1 - 1, 1) + \lambda_0 \Delta_k^A(m_1, 1) + \lambda_3 [(1 - (\lambda_1 + \lambda_2)^{m_1 - 1}) - (1 - (\lambda_1 + \lambda_2)^{m_1})] \\ &= \lambda_3 + \lambda_1 \Delta_k^A(m_1 - 1, 1) + \lambda_0 \Delta_k^A(m_1, 1) - \lambda_3 (\lambda_3 + \lambda_0) (\lambda_1 + \lambda_2)^{m_1 - 1}, \end{aligned}$$

which also increases as m_1 increases by the induction hypothesis. Thus, cases (1a) though (1d) shows that $\Delta_n^A(\mathbf{m})$ increases in m_1 for $n \geq 1$ and $\mathbf{m} = (m_1, 1)$, $m_1 \geq 1$.

Case (3): $\mathbf{m} = (1, m_2)$, $m_2 \geq 2$. We want to show that

$$\Delta_{k+1}^A(2, m_2) - \Delta_{k+1}^A(1, m_2) = [V_{k+1}(1, m_2) - V_{k+1}(2, m_2 - 1)] - [V_{k+1}(0, m_2) - V_{k+1}(1, m_2 - 1)] \geq 0.$$

Again, we separate into a few subcases. If the optimal action at state $(1, m_2 - 1)$ is $\{1\}$ - $\{2\}$, then $\Delta_k^A(1, m_2 - 1) \geq 0$. It follows that $\Delta_k^A(2, m_2 - 1) \geq 0$ by the induction hypothesis, and the optimal action at state $(2, m_2 - 1)$ is also $\{1\}$ - $\{2\}$. But the optimal actions at state $(1, m_2)$ can still be either $\{1\}$ - $\{2\}$ or $\{2\}$ - $\{1\}$. Following this logic, we need to consider four subcases. Case (3a): the optimal actions at state $(1, m_2 - 1)$, $(2, m_2 - 1)$ and $(1, m_2)$ are all $\{1\}$ - $\{2\}$. Case (3b): the optimal actions at state $(1, m_2 - 1)$, $(2, m_2 - 1)$ and $(1, m_2)$ are $\{1\}$ - $\{2\}$, $\{1\}$ - $\{2\}$ and $\{2\}$ - $\{1\}$, respectively. Case (3c): the optimal actions at state $(1, m_2 - 1)$, $(2, m_2 - 1)$ and $(1, m_2)$ are all $\{2\}$ - $\{1\}$. Case (3d): the optimal actions at state $(1, m_2 - 1)$, $(2, m_2 - 1)$ and $(1, m_2)$ are $\{2\}$ - $\{1\}$, $\{1\}$ - $\{2\}$ and $\{2\}$ - $\{1\}$, respectively.

For case (3a), we have

$$\begin{aligned} &\Delta_{k+1}^A(2, m_2) - \Delta_{k+1}^A(1, m_2) \\ &= \lambda_1 [1 - V_k(0, m_2 - 1)] + (\lambda_1 + \lambda_2) \Delta_k^A(1, m_2) \\ &\quad + \lambda_3 [\Delta_k^A(2, m_2 - 1) - \Delta_k^A(1, m_2 - 1)] + \lambda_0 [\Delta_k^A(2, m_2) - \Delta_k^A(1, m_2)] \geq 0, \end{aligned}$$

where the inequality follows from the fact that the first term is trivially nonnegative, the second term is positive as the optimal actions at state $(1, m_2)$ is $\{1\}$ - $\{2\}$ and the last two terms are nonnegative by the induction hypothesis.

For case (3b), we have

$$\begin{aligned} & \Delta_{k+1}^A(2, m_2) - \Delta_{k+1}^A(1, m_2) \\ &= \lambda_1[-V_k(1, m_2 - 1) + 1 + V_k(0, m_2 - 1)] \\ & \quad + \lambda_3[\Delta_k^A(2, m_2 - 1) - \Delta_k^A(1, m_2 - 1)] + \lambda_0[\Delta_k^A(2, m_2) - \Delta_k^A(1, m_2)] \geq 0, \end{aligned}$$

where the first term is nonnegative following Lemma 1 and the other two terms are nonnegative following the induction hypothesis.

For case (3c), we have

$$\begin{aligned} & \Delta_{k+1}^A(2, m_2) - \Delta_{k+1}^A(1, m_2) \\ &= \lambda_1[-V_k(1, m_2 - 1) + 1 + V_k(0, m_2 - 1)] \\ & \quad + (\lambda_2 + \lambda_3)[\Delta_k^A(2, m_2 - 1) - \Delta_k^A(1, m_2 - 1)] + \lambda_0[\Delta_k^A(2, m_2) - \Delta_k^A(1, m_2)] \geq 0, \end{aligned}$$

following a similar argument of case (3b).

For case (3d), we have

$$\begin{aligned} & \Delta_{k+1}^A(2, m_2) - \Delta_{k+1}^A(1, m_2) \\ &= \lambda_1[-V_k(1, m_2 - 1) + 1 + V_k(0, m_2 - 1)] \\ & \quad - \lambda_2\Delta_k^A(1, m_2 - 1) + \lambda_3[\Delta_k^A(2, m_2 - 1) - \Delta_k^A(1, m_2 - 1)] + \lambda_0[\Delta_k^A(2, m_2) - \Delta_k^A(1, m_2)] \geq 0, \end{aligned}$$

following a similar logic of case (3b) and the fact that $\Delta_k^A(1, m_2 - 1) \leq 0$ (because the optimal action at state $(1, m_2 - 1)$ is $\{2\}$ - $\{1\}$). This completes the proof of case (3).

For case (4) $\mathbf{m} \geq (2, 2)$, we evaluate $V_n(\mathbf{m} - \mathbf{e}_1) - V_n(\mathbf{m} - \mathbf{e}_2)$ and need to consider four subcases. Case (4a): if the optimal actions at states $(\mathbf{m} - \mathbf{e}_1)$ and $(\mathbf{m} - \mathbf{e}_2)$ are both $\{1\}$ - $\{2\}$. Then

$$\begin{aligned} & \Delta_{k+1}^A(\mathbf{m}) = V_{k+1}(\mathbf{m} - \mathbf{e}_1) - V_{k+1}(\mathbf{m} - \mathbf{e}_2) \\ &= (\lambda_1 + \lambda_2)\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + \lambda_3\Delta_k^A(\mathbf{m} - \mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}), \end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4b): if the optimal actions at states $(\mathbf{m} - \mathbf{e}_1)$ and $(\mathbf{m} - \mathbf{e}_2)$ are both $\{2\}$ - $\{1\}$. Then,

$$\begin{aligned} & \Delta_{k+1}^A(\mathbf{m}) = V_{k+1}(\mathbf{m} - \mathbf{e}_1) - V_{k+1}(\mathbf{m} - \mathbf{e}_2) \\ &= \lambda_1\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + (\lambda_2 + \lambda_3)\Delta_k^A(\mathbf{m} - \mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}), \end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4c): if the optimal actions at states $(\mathbf{m} - \mathbf{e}_1)$ and $(\mathbf{m} - \mathbf{e}_2)$ are $\{1\}$ - $\{2\}$ and $\{2\}$ - $\{1\}$, respectively. Then,

$$\begin{aligned} & \Delta_{k+1}^A(\mathbf{m}) = V_{k+1}(\mathbf{m} - \mathbf{e}_1) - V_{k+1}(\mathbf{m} - \mathbf{e}_2) \\ &= \lambda_1\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + \lambda_2[V_k(\mathbf{m} - 2\mathbf{e}_1) - V_k(\mathbf{m} - 2\mathbf{e}_2)] + \lambda_3\Delta_k^A(\mathbf{m} - \mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}) \\ &= \lambda_1\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + \lambda_2[\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + \Delta_k^A(\mathbf{m} - \mathbf{e}_2)] + \lambda_3\Delta_k^A(\mathbf{m} - \mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}), \end{aligned}$$

which increases in m_1 by the induction hypothesis. Case (4d): if the optimal actions at states $(\mathbf{m} - \mathbf{e}_1)$ and $(\mathbf{m} - \mathbf{e}_2)$ are $\{2\}$ - $\{1\}$ and $\{1\}$ - $\{2\}$, respectively. Then,

$$\Delta_{k+1}^A(\mathbf{m}) = V_{k+1}(\mathbf{m} - \mathbf{e}_1) - V_{k+1}(\mathbf{m} - \mathbf{e}_2) = \lambda_1\Delta_k^A(\mathbf{m} - \mathbf{e}_1) + \lambda_3\Delta_k^A(\mathbf{m} - \mathbf{e}_2) + \lambda_0\Delta_k^A(\mathbf{m}),$$

which increases in m_1 by the induction hypothesis.

Finally, we show our claim (92), which can be easily done by induction. When $k = 1$, $2V_k(1, 0) - V_k(2, 0) = 2(\lambda_1 + \lambda_2) - [1 - (1 - \lambda_1 - \lambda_2)^2] = (\lambda_1 + \lambda_2)^2 \geq 0$. Assume this holds up to $k = u$. Consider $k = u + 1$. We

have that

$$\begin{aligned}
& 2V_{u+1}(1, 0) - V_{u+1}(2, 0) \\
&= 2[(\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_0)V_u(1, 0)] - [(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2)V_u(1, 0) + (\lambda_3 + \lambda_0)V_u(2, 0)] \\
&= (\lambda_1 + \lambda_2)[1 - V_u(1, 0)] + (\lambda_3 + \lambda_0)[2V_u(1, 0) - V_u(2, 0)] \geq 0,
\end{aligned}$$

proving our claim (92) and completing the whole proof. \square \square